Lab 7: Inverses and Projections Solutions

EECS 245, Spring 2026 at the University of Michigan

due for completion at 11:59PM Ann Arbor Time on Monday, June 1st, 2026

Name: ______________________________

uniqname: ______________________________

Each lab worksheet will contain several activities, some of which will involve writing code and others that will involve writing math on paper. To receive credit for a lab, you must complete as many of the activities as you can in 2 hours and submit a PDF of your work to Gradescope. We will provide specific instructions on how to submit programming activities (e.g. submitting the notebook or including a screenshot of some output).

Recap: Inverses (Chapter 6.2) and Linear Transformations (Chapter 6.1)

(We provide a recap of projections and the normal equation in Activity 3.)

- An $n \times n$ square matrix A is **invertible** if and only if $\text{rank}(A) = n$, which also means that A's columns are linearly independent (along with several other equivalent conditions).
- If A is invertible, then its **inverse** A^{-1} is the **unique** $n \times n$ matrix such that $AA^{-1} = I = A^{-1}A$.
- The determinant of a square matrix A is the volume of the n-dimensional cube with side length 1 after it is transformed by A.
  - If $\text{det}(A) = 0$, then A is not invertible.
  - If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\text{det}(A) = ad - bc$ and $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.
  - The determinant satisfies several properties, including that $\text{det}(AB) = \text{det}(A)\text{det}(B)$ and $\text{det}(A^T) = \text{det}(A)$.
- A linear transformation is a function $\mathbb{R}^d \to \mathbb{R}^n$ such that

$$f(\vec{x} + \vec{y}) = f(\vec{x}) + f(\vec{y}), \qquad f(c\vec{x}) = cf(\vec{x})$$

  - Every linear transformation has a corresponding $n \times d$ matrix A where $f(\vec{x}) = A\vec{x}$.
  - If A is square, then $f(\vec{x}) = A\vec{x}$ is a function from $\mathbb{R}^n$ to $\mathbb{R}^n$, and A is invertible if and only if the function f is invertible.

Activity 1: PrairieLearn Practice Problems

We're testing out a new website for practicing linear algebra problems: PrairieLearn.

Click **this link** to access the relevant problems for this activity. It consists of 6 problems, each worth 1 point. The numbers in the problems are randomized; everyone will receive slightly different problems.

To get credit for Activity 1, you must **eventually** correctly answer all 6 problems, earning a score of 6/6. If you answer a problem incorrectly, click "New Variant" to generate a new version and then try again. There is no penalty for answering a problem incorrectly, as long as you eventually get it correct.

To be clear, you don't need to include anything in your PDF for Activity 1; we will manually verify that you've finished all 6 problems.

If you have trouble accessing PrairieLearn, message Suraj on Slack.

Activity 2: Linear Transformations

Suppose $f : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation represented by the matrix A.

Furthermore, suppose that $f\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}0\\3\\4\end{bmatrix}$, $f\left(\begin{bmatrix}0\\10\\0\end{bmatrix}\right) = \begin{bmatrix}0\\4\\-3\end{bmatrix}$, and $f\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}1\\0\\0\end{bmatrix}$.

a) Find $f\left(\begin{bmatrix}2\\1\\2\end{bmatrix}\right)$. **After that**, find the matrix A corresponding to f, i.e. where $f(\vec{x}) = A\vec{x}$.

Solution: First, using just the properties of linear transformations, we can find $f\left(\begin{bmatrix}2\\1\\2\end{bmatrix}\right)$. Recall that a linear transformation f satisfies

- $f(\vec{x}+\vec{y}) = f(\vec{x}) + f(\vec{y})$
- $f(c\vec{x}) = cf(\vec{x})$

In other words, f preserves linear combinations, i.e. $f(a\vec{x}+b\vec{y}) = af(\vec{x}) + bf(\vec{y})$.

Let's decompose $f\left(\begin{bmatrix}2\\1\\2\end{bmatrix}\right)$ into a linear combination of outputs we already know.

$$\begin{aligned}
f\left(\begin{bmatrix}2\\1\\2\end{bmatrix}\right) &= f\left(2\begin{bmatrix}1\\0\\0\end{bmatrix} + \frac{1}{10}\begin{bmatrix}0\\10\\0\end{bmatrix} + 2\begin{bmatrix}0\\0\\1\end{bmatrix}\right)\\
&= \underbrace{2f\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) + \frac{1}{10}f\left(\begin{bmatrix}0\\10\\0\end{bmatrix}\right) + 2f\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)}_{\text{property of linear transformations}}\\
&= 2\begin{bmatrix}0\\3\\4\end{bmatrix} + \frac{1}{10}\begin{bmatrix}0\\4\\-3\end{bmatrix} + 2\begin{bmatrix}1\\0\\0\end{bmatrix}\\
&= \begin{bmatrix}0\\6\\8\end{bmatrix} + \begin{bmatrix}0\\4/10\\-3/10\end{bmatrix} + \begin{bmatrix}2\\0\\0\end{bmatrix}\\
&= \begin{bmatrix}2\\6.4\\7.7\end{bmatrix}
\end{aligned}$$

So, $f\left(\begin{bmatrix}2\\1\\2\end{bmatrix}\right) = \begin{bmatrix}2\\6.4\\7.7\end{bmatrix}$.

Next, we need to find the matrix A corresponding to f, i.e. where $f(\vec{x}) = A\vec{x}$. Since f is a linear transformation, we know that $f(\vec{x}) = A\vec{x}$ for some matrix A.

- The first column of A is given by $f\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)$, which we're told is $\begin{bmatrix}0\\3\\4\end{bmatrix}$.
- The second column of A is given by $f\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)$, which we've computed above to be $\begin{bmatrix}0\\4/10\\-3/10\end{bmatrix}$.

- The third column of A is given by $f\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)$, which we're told is $\begin{bmatrix}1\\0\\0\end{bmatrix}$.

So,

$$A = \begin{bmatrix}0 & 0 & 1\\3 & 4/10 & 0\\4 & -3/10 & 0\end{bmatrix}$$

If that seems too simple to be true, you can verify that multiplying A by $\begin{bmatrix}2\\1\\2\end{bmatrix}$ gives us $\begin{bmatrix}2\\6.4\\7.7\end{bmatrix}$, and that multiplying A by the three vectors in the question give the outputs we're told. Remember that $A\begin{bmatrix}1\\0\\0\end{bmatrix}$ returns just the first column of A, and so on.

b) Find a **diagonal** matrix D and an **orthogonal** matrix Q such that $A = QD$. (Not every matrix can be written in this form, but this particular A can.) Then, describe **in English** how f transforms a vector $\vec{x}$.

Solution:
Remember that

- Diagonal matrices have 0s everywhere except on the diagonal, and have the effect of stretching/compressing each axis/dimension of the input vector independently.
- Orthogonal matrices have columns that are orthonormal, meaning their columns are unit vectors that are orthogonal to one another.

In

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 3 & 4/10 & 0 \\ 4 & -3/10 & 0 \end{bmatrix}$$

You might notice that A's first column is 5 times $\begin{bmatrix} 0 \\ 3/5 \\ 4/5 \end{bmatrix}$, which is a unit vector. You might also notice that A's second column is $(1/2)$ times $\begin{bmatrix} 0 \\ 4/5 \\ -3/5 \end{bmatrix}$, which is another unit vector orthogonal to the first column. And finally, A's third column is already a unit vector, and its orthogonal to the first two columns.

So, if we put these orthogonal unit vectors into the columns of Q, and the scaling factors of 5, $1/2$, and 1 into the diagonal of D, we have

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 3 & 4/10 & 0 \\ 4 & -3/10 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 3/5 & 4/5 & 0 \\ 4/5 & -3/5 & 0 \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} 5 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{D}$$

So, $Q = \begin{bmatrix} 0 & 0 & 1 \\ 3/5 & 4/5 & 0 \\ 4/5 & -3/5 & 0 \end{bmatrix}$ and $D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$f(\vec{x}) = QD\vec{x}$ transforms f by first scaling $\vec{x}$'s first component by 5, second component by 1/2, and leaving the third component as-is, and then rotating it by the orthogonal matrix Q.

Orthogonal matrices rotate the vectors that they're multiplied by. In $\mathbb{R}^2$, this is a rotation by some angle θ. It's harder to describe rotations in $\mathbb{R}^3$ and beyond, but the key property that describes the effect of an orthogonal matrix Q is that for any vector $\vec{x}$,

$$\|Q\vec{x}\| = \|\vec{x}\|$$

as you proved in Homework 6, meaning that all an orthogonal matrix is doing is changing the angle of the vector, not its length.

So, $f(\vec{x})$ first scales $\vec{x}$'s components, and then rotates the resulting vector.

c) Using your $A = QD$ decomposition from part **b)**, find A^{-1}.

Hint: Recall that for orthogonal matrices, $QQ^T = Q^TQ = I$. And, for any invertible matrices A and B, $(AB)^{-1} = B^{-1}A^{-1}$.

Solution:
Since $A = QD$, we have

$$A^{-1} = D^{-1}Q^{-1}$$

Since D is diagonal, its inverse is just the diagonal matrix with the reciprocal of each diagonal entry. This corresponds to "unstretching" the vector in each dimension. So,

$$D^{-1} = \begin{bmatrix} 1/5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

And, since Q is orthogonal, we know $Q^TQ = I$, meaning that $Q^{-1} = Q^T$, i.e. Q's inverse is its transpose. This corresponds to "undoing" the rotation of the vector.

$$Q^{-1} = Q^T = \begin{bmatrix} 0 & 3/5 & 4/5 \\ 0 & 4/5 & -3/5 \\ 1 & 0 & 0 \end{bmatrix}$$

Putting these building blocks together gives us

$$A^{-1} = D^{-1}Q^{-1} = D^{-1}Q^T = \begin{bmatrix} 1/5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 3/5 & 4/5 \\ 0 & 4/5 & -3/5 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 3/25 & 4/25 \\ 0 & 8/5 & -6/5 \\ 1 & 0 & 0 \end{bmatrix}$$

d) Given the English definition of f from part **b) alone**, find $\det(A)$. (You can verify your work using the formula in **Chapter 6.1**.)

Solution: Recall, $f(\vec{x})$ first scales $\vec{x}$'s first component by 5, second component by $1/2$, and leaves the third component as-is, and then it rotates the resulting vector in a way that preserves its length.

Intuitively, $\det(A)$ should be the product of the scaling factors, i.e. $5 \cdot 1/2 \cdot 1 = 5/2$.

Activity 3: Projecting onto the Column Space

Note: We've recorded a **YouTube playlist** walking through the activities in this lab.

Suppose X is an $n \times d$ matrix with columns $\vec{x}^{(1)}, \vec{x}^{(2)}, \dots, \vec{x}^{(d)}$ and $\vec{y} \in \mathbb{R}^n$. Then, the projection of $\vec{y}$ onto colsp(X) is the vector

$$\vec{p} = X\vec{w}^* = w_1^*\vec{x}^{(1)} + w_2^*\vec{x}^{(2)} + \cdots + w_d^*\vec{x}^{(d)}$$

where $\vec{w}^* \in \mathbb{R}^d$ is chosen to satisfy the **normal equation**,

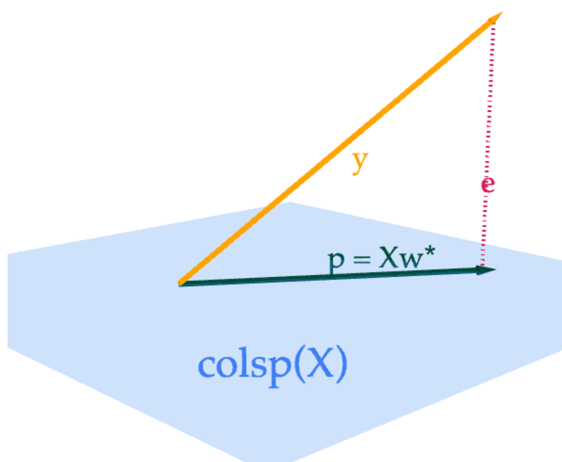
$$X^TX\vec{w} = X^T\vec{y}$$

If X's columns are linearly independent, $\vec{w}^*$ is the unique vector

$$\vec{w}^* = (X^TX)^{-1}X^T\vec{y}$$

Of all vectors in colsp(X), $X\vec{w}^*$ is the one that is closest to $\vec{y}$, meaning it minimizes

$$\|\vec{y} - X\vec{w}\|^2$$



As we will see in Tuesday's lecture, $\vec{w}^*$ contains the **optimal model parameters** for linear regression, when we fill our X (carefully) with our input variables and $\vec{y}$ with our output variables.

a) Let $X = \begin{bmatrix} 2 & 1 \\ 0 & -3 \\ 0 & 0 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$. Find $\vec{w}^*$, $\vec{p}$, and $\vec{e} = \vec{y} - \vec{p}$, and verify that $\vec{e}$ is orthogonal to colsp(X) by showing that it is orthogonal to each of X's columns.

Solution: For

$$X = \begin{bmatrix} 2 & 1 \\ 0 & -3 \\ 0 & 0 \end{bmatrix} \qquad \vec{y} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

we begin by computing the normal equation components:

$$X^T X = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -3 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & -3 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 10 \end{bmatrix}$$

The determinant of $X^T X$ is 36, so

$$(X^T X)^{-1} = \frac{1}{36} \begin{bmatrix} 10 & -2 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} \frac{5}{18} & -\frac{1}{18} \\ -\frac{1}{18} & \frac{1}{9} \end{bmatrix}$$

Next, compute

$$X^T \vec{y} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -3 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ -7 \end{bmatrix}$$

Therefore,

$$\vec{w}^* = (X^T X)^{-1} X^T \vec{y} = \begin{bmatrix} \frac{5}{18} & -\frac{1}{18} \\ -\frac{1}{18} & \frac{1}{9} \end{bmatrix} \begin{bmatrix} 4 \\ -7 \end{bmatrix} = \begin{bmatrix} \frac{27}{18} \\ -1 \end{bmatrix} = \begin{bmatrix} 1.5 \\ -1 \end{bmatrix}$$

Now compute the projection:

$$\vec{p} = X\vec{w}^* = \begin{bmatrix} 2 & 1 \\ 0 & -3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1.5 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$$

The error vector is:

$$\vec{e} = \vec{y} - \vec{p} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$$

To verify orthogonality:

$$X^T \vec{e} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -3 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus, $\vec{e}$ is orthogonal to both columns of X as required.

b) Find scalars a and b such that $a\begin{bmatrix}2\\0\\0\end{bmatrix}+b\begin{bmatrix}1\\-3\\0\end{bmatrix}$ is as close as possible to $\begin{bmatrix}1\\9\\2\end{bmatrix}$. *Hint: You can reuse most of your work from part **a)**.*

c) Now, suppose $X=\begin{bmatrix}1 & 1\\1 & -1\\1 & 0\end{bmatrix}$ and $\vec{y}=\begin{bmatrix}2\\3\\4\end{bmatrix}$. We've already computed $\vec{w}^*$, $\vec{p}$, and $\vec{e}$ here:

$$\vec{w}^*=\begin{bmatrix}3\\-\frac{1}{2}\end{bmatrix},\quad \vec{p}=\begin{bmatrix}5/2\\7/2\\3\end{bmatrix},\quad \vec{e}=\vec{y}-\vec{p}=\begin{bmatrix}-1/2\\-1/2\\1\end{bmatrix}$$

Notice that the components of this $\vec{e}$ add up to 0, but this doesn't happen with your $\vec{e}$ from part **a)**. **Why?** *Hint: The answer is not that $\vec{y}$ is in colsp(X) — it isn't in part **a)** and it isn't here either. Rather, it has something to do with the difference between the two X's. This is a hugely important result, and one that will 100% appear on Midterm 2.*

Solution: In both parts, $X^T\vec{e}=\begin{bmatrix}0\\0\end{bmatrix}$, but only in this new example is the sum of the components in $\vec{e}$ exactly 0 (in part **a)**, it's 4).

This has to do with the fact that in each case, $\vec{e}$ is orthogonal to **any linear combination of the columns of** X. This is what it means for $\vec{e}$ to be orthogonal to colsp(X).

In subpart (ii), one of the columns of X is $\begin{bmatrix}1\\1\\1\end{bmatrix}$, meaning that $\vec{e}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}=e_1+e_2+e_3=0$. However, no linear combination of the columns of X in subpart (i) gives a vector of $\begin{bmatrix}1\\1\\1\end{bmatrix}$, so we can't use this logic to guarantee the error vector's components sum to 0 in subpart (i). (I say "guarantee" because while the error vector's components don't sum to 0 for $\vec{y}=\begin{bmatrix}2\\3\\4\end{bmatrix}$, they still could for some other $\vec{y}$ projected onto the column space of X in subpart (i).)

Don't forget this fact: the existence of a column of 1's in X (or in X's column space) guarantees that the error vector $\vec{e}$ when projecting $\vec{y}$ onto colsp(X) will have a sum of components equal to 0, no matter what the other columns of X are and no matter what $\vec{y}$ is.

Taking another look at the formula $\vec{p} = X\vec{w}^*$, we see that it's equivalent to

$$\vec{p} = X\vec{w}^* = X(X^TX)^{-1}X^T\vec{y} = P\vec{y}$$

where $P = X(X^TX)^{-1}X^T$ is called the **projection matrix**, discussed in **Chapter 6.4**. Multiplying $P\vec{y}$ is equivalent to projecting $\vec{y}$ onto colsp(X).

d) Recall that X is an $n \times d$ matrix (meaning it's not necessarily square), which makes $P = X(X^TX)^{-1}X^T$ an $n \times n$ matrix.

Fill in the blanks: X^TX is invertible if and only if X's columns are ____.

> **Solution:** X^TX is invertible if and only if X's columns are **linearly independent**. This is because rank(X) = rank(X^TX), as we proved in **Chapter 5.4**, and a matrix is invertible if and only if its rank is equal to its number of columns.
>
> For rank$(X^TX) = d$, we need rank$(X) = d$, meaning X's columns must be linearly independent.

e) In this part only, suppose X is an $n \times 1$ matrix, i.e. it is a vector. Then,

(i) What is the value of $\vec{w}^*$, and how does it relate to what we learned in **Chapter 3.4**? (What type of object is $(X^TX)^{-1}$ when X is a vector?)

(ii) What is the value of the matrix P, and how does it relate to what we learned in **Homework 6, Problem 5**?

Solution: Suppose X is the vector $\vec{x}$. Remember that $\vec{x}^T\vec{x}$ is a scalar; it's the dot product of $\vec{x}$ with itself. So, $(X^TX)^{-1} = (\vec{x}^T\vec{x})^{-1} = \frac{1}{\vec{x}^T\vec{x}}$, since the inverse of a scalar is the reciprocal of that scalar.

(i)

$$\vec{w}^* = (X^TX)^{-1}X^T\vec{y} = (\vec{x}^T\vec{x})^{-1}\vec{x}^T\vec{y} = \frac{\vec{x}^T\vec{y}}{\vec{x}^T\vec{x}} = \frac{\vec{x}\cdot\vec{y}}{\vec{x}\cdot\vec{x}}$$

This is the same formula for the optimal value of k from Chapter 3.4, where we approximated $\vec{y}$ by a scalar multiple of $\vec{x}$, called $k\vec{x}$.

(ii)

$$P = X(X^TX)^{-1}X^T = \vec{x}(\vec{x}^T\vec{x})^{-1}\vec{x}^T = \frac{\vec{x}\vec{x}^T}{\vec{x}^T\vec{x}}$$

In **Homework 6, Problem 5**, we found that the matrix P that projects $\vec{y}$ onto the line spanned by the **unit vector** $\vec{x}$ is given by

$$P = \begin{bmatrix} x_1^2 & x_1x_2 \\ x_1x_2 & x_2^2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T = \vec{x}\vec{x}^T$$

Here, $\vec{x}$ isn't a unit vector, hence the division by $\vec{x}^T\vec{x}$.

$$P = X(X^TX)^{-1}X^T$$

f) Show that P is both symmetric (meaning that $P^T = P$) and idempotent (meaning that $P^2 = P$). Then, explain in English how P's idempotence relates to the linear transformation of projecting $\vec{y}$ onto $\text{colsp}(X)$.

Solution: To show that P is **symmetric**, we need to show that $P^T = P$. Recall that $(AB)^T = B^TA^T$, so

$$\begin{aligned}
P^T &= \left(X(X^TX)^{-1}X^T\right)^T \\
&= \left(X^T\right)^T \left((X^TX)^{-1}\right)^T X^T \\
&= X(X^TX)^{-1}X^T \\
&= P
\end{aligned}$$

To go from line 2 to line 3, we used the fact that X^TX is symmetric, so $(X^TX)^{-1}$ is also symmetric. Remember that X^TX contains the dot products of all pairs of X's columns.

To show that P is **idempotent**, we need to show that $P^2 = P$.

$$\begin{aligned}
P^2 &= \left(X(X^TX)^{-1}X^T\right)\left(X(X^TX)^{-1}X^T\right) \\
&= X(X^TX)^{-1}\left(X^TX\right)(X^TX)^{-1}X^T \\
&= X(X^TX)^{-1}IX^T \\
&= X(X^TX)^{-1}X^T \\
&= P
\end{aligned}$$

The fact that P is idempotent means that applying P twice (or three times, or four times, etc.) to a vector is the same as applying it once. Once we've already projected $\vec{y}$ onto $\text{colsp}(X)$, we don't need to project it again, since the projection $\vec{p} = P\vec{y}$ is already in $\text{colsp}(X)$.

g) In the rare case that X is an $n \times n$ square matrix, and $\text{rank}(X) = n$, what is P? What does this say about the relationship between $\vec{y}$, $\vec{p}$, and $\text{colsp}(X)$? *Hint: Use the fact that* $(AB)^{-1} = B^{-1}A^{-1}$.

Solution: If X is an $n \times n$ square matrix, and $\text{rank}(X) = n$, then X is invertible, and so is X^T (since if X has n linearly independent columns, it must have n linearly independent rows).

$(X^T X)^{-1} = X^{-1}(X^T)^{-1}$ because $(AB)^{-1} = B^{-1}A^{-1}$.

Then,

$$P = X(X^T X)^{-1}X^T = \underbrace{XX^{-1}}_{I}\underbrace{(X^T)^{-1}X^T}_{I} = I$$

So, if X is an $n \times n$ square matrix with $\text{rank}(X) = n$, then $P = I$. What this says is that $\text{colsp}(X) = \mathbb{R}^n$, meaning that any vector $\vec{y} \in \mathbb{R}^n$ can be represented as a linear combination of X's columns, so the projection of $\vec{y}$ onto $\text{colsp}(X)$ is just $\vec{y}$ itself, i.e. $\vec{p} = P\vec{y} = \vec{y}$.