Lab 7: Inverses and Projections

EECS 245, Spring 2026 at the University of Michigan

due for completion at 11:59PM Ann Arbor Time on Monday, June 1st, 2026

Name: ____________________

uniqname: ____________________

Each lab worksheet will contain several activities, some of which will involve writing code and others that will involve writing math on paper. To receive credit for a lab, you must complete as many of the activities as you can in 2 hours and submit a PDF of your work to Gradescope. We will provide specific instructions on how to submit programming activities (e.g. submitting the notebook or including a screenshot of some output).

Recap: Inverses (Chapter 6.2) and Linear Transformations (Chapter 6.1)

(We provide a recap of projections and the normal equation in Activity 3.)

- An $n \times n$ square matrix A is **invertible** if and only if $\text{rank}(A) = n$, which also means that A's columns are linearly independent (along with several other equivalent conditions).
- If A is invertible, then its **inverse** A^{-1} is the **unique** $n \times n$ matrix such that $AA^{-1} = I = A^{-1}A$.
- The determinant of a square matrix A is the volume of the n-dimensional cube with side length 1 after it is transformed by A.
  - If $\text{det}(A) = 0$, then A is not invertible.
  - If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\text{det}(A) = ad - bc$ and $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.
  - The determinant satisfies several properties, including that $\text{det}(AB) = \text{det}(A)\text{det}(B)$ and $\text{det}(A^T) = \text{det}(A)$.
- A linear transformation is a function $\mathbb{R}^d \to \mathbb{R}^n$ such that

$$f(\vec{x} + \vec{y}) = f(\vec{x}) + f(\vec{y}), \qquad f(c\vec{x}) = cf(\vec{x})$$

  - Every linear transformation has a corresponding $n \times d$ matrix A where $f(\vec{x}) = A\vec{x}$.
  - If A is square, then $f(\vec{x}) = A\vec{x}$ is a function from $\mathbb{R}^n$ to $\mathbb{R}^n$, and A is invertible if and only if the function f is invertible.

Activity 1: PrairieLearn Practice Problems

We're testing out a new website for practicing linear algebra problems: PrairieLearn.

Click **this link** to access the relevant problems for this activity. It consists of 6 problems, each worth 1 point. The numbers in the problems are randomized; everyone will receive slightly different problems.

To get credit for Activity 1, you must **eventually** correctly answer all 6 problems, earning a score of 6/6. If you answer a problem incorrectly, click "New Variant" to generate a new version and then try again. There is no penalty for answering a problem incorrectly, as long as you eventually get it correct.

To be clear, you don't need to include anything in your PDF for Activity 1; we will manually verify that you've finished all 6 problems.

If you have trouble accessing PrairieLearn, message Suraj on Slack.

Activity 2: Linear Transformations

Suppose $f : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation represented by the matrix A.

Furthermore, suppose that $f\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}0\\3\\4\end{bmatrix}$, $f\left(\begin{bmatrix}0\\10\\0\end{bmatrix}\right) = \begin{bmatrix}0\\4\\-3\end{bmatrix}$, and $f\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}1\\0\\0\end{bmatrix}$.

a) Find $f\left(\begin{bmatrix}2\\1\\2\end{bmatrix}\right)$. **After that**, find the matrix A corresponding to f, i.e. where $f(\vec{x}) = A\vec{x}$.

b) Find a **diagonal** matrix D and an **orthogonal** matrix Q such that $A = QD$. (Not every matrix can be written in this form, but this particular A can.) Then, describe **in English** how f transforms a vector $\vec{x}$.

c) Using your $A = QD$ decomposition from part **b)**, find A^{-1}.

Hint: Recall that for orthogonal matrices, $QQ^T = Q^TQ = I$. *And, for any invertible matrices* A *and* B, $(AB)^{-1} = B^{-1}A^{-1}$.

d) Given the English definition of f from part **b) alone**, find $\det(A)$. (You can verify your work using the formula in **Chapter 6.1**.)

Activity 3: Projecting onto the Column Space

Note: We've recorded a **YouTube playlist** walking through the activities in this lab.

Suppose X is an $n \times d$ matrix with columns $\vec{x}^{(1)}, \vec{x}^{(2)}, \ldots, \vec{x}^{(d)}$ and $\vec{y} \in \mathbb{R}^n$. Then, the projection of $\vec{y}$ onto colsp(X) is the vector

$$\vec{p} = X\vec{w}^* = w_1^*\vec{x}^{(1)} + w_2^*\vec{x}^{(2)} + \cdots + w_d^*\vec{x}^{(d)}$$

where $\vec{w}^* \in \mathbb{R}^d$ is chosen to satisfy the **normal equation**,

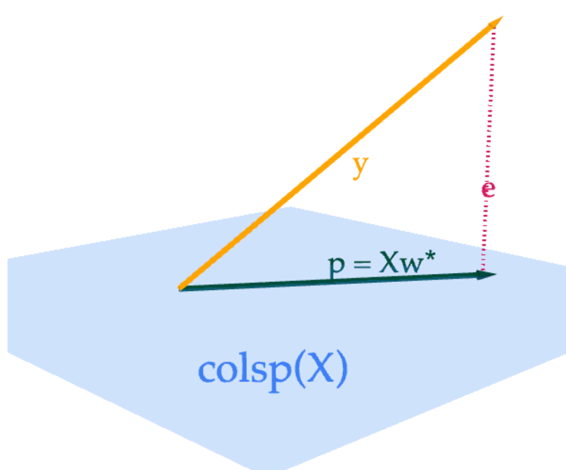
$$X^TX\vec{w} = X^T\vec{y}$$

If X's columns are linearly independent, $\vec{w}^*$ is the unique vector

$$\vec{w}^* = (X^TX)^{-1}X^T\vec{y}$$

Of all vectors in colsp(X), $X\vec{w}^*$ is the one that is closest to $\vec{y}$, meaning it minimizes

$$\|\vec{y} - X\vec{w}\|^2$$



As we will see in Tuesday's lecture, $\vec{w}^*$ contains the **optimal model parameters** for linear regression, when we fill our X (carefully) with our input variables and $\vec{y}$ with our output variables.

a) Let $X = \begin{bmatrix} 2 & 1 \\ 0 & -3 \\ 0 & 0 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$. Find $\vec{w}^*$, $\vec{p}$, and $\vec{e} = \vec{y} - \vec{p}$, and verify that $\vec{e}$ is orthogonal to colsp(X) by showing that it is orthogonal to each of X's columns.

b) Find scalars a and b such that $a \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$ is as close as possible to $\begin{bmatrix} 1 \\ 9 \\ 2 \end{bmatrix}$. *Hint: You can reuse most of your work from part* ***a)****.*

c) Now, suppose $X = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$. We've already computed $\vec{w}^*$, $\vec{p}$, and $\vec{e}$ here:

$$\vec{w}^* = \begin{bmatrix} 3 \\ -\frac{1}{2} \end{bmatrix}, \quad \vec{p} = \begin{bmatrix} 5/2 \\ 7/2 \\ 3 \end{bmatrix}, \quad \vec{e} = \vec{y} - \vec{p} = \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$$

Notice that the components of this $\vec{e}$ add up to 0, but this doesn't happen with your $\vec{e}$ from part **a)**. **Why?** *Hint: The answer is not that $\vec{y}$ is in colsp(X) — it isn't in part* ***a)*** *and it isn't here either. Rather, it has something to do with the difference between the two X's. This is a hugely important result, and one that will 100% appear on Midterm 2.*

Taking another look at the formula $\vec{p} = X\vec{w}^*$, we see that it's equivalent to

$$\vec{p} = X\vec{w}^* = X(X^TX)^{-1}X^T\vec{y} = P\vec{y}$$

where $P = X(X^TX)^{-1}X^T$ is called the **projection matrix**, discussed in **Chapter 6.4**. Multiplying $P\vec{y}$ is equivalent to projecting $\vec{y}$ onto $\text{colsp}(X)$.

d) Recall that X is an $n \times d$ matrix (meaning it's not necessarily square), which makes $P = X(X^TX)^{-1}X^T$ an $n \times n$ matrix.

Fill in the blanks: X^TX is invertible if and only if X's columns are ____.

e) In this part only, suppose X is an $n \times 1$ matrix, i.e. it is a vector. Then,

(i) What is the value of $\vec{w}^*$, and how does it relate to what we learned in **Chapter 3.4**? (What type of object is $(X^TX)^{-1}$ when X is a vector?)

(ii) What is the value of the matrix P, and how does it relate to what we learned in **Homework 6, Problem 5**?

$$P = X(X^T X)^{-1} X^T$$

f) Show that P is both symmetric (meaning that $P^T = P$) and idempotent (meaning that $P^2 = P$). Then, explain in English how P's idempotence relates to the linear transformation of projecting $\vec{y}$ onto colsp(X).

g) In the rare case that X is an $n \times n$ square matrix, and rank$(X) = n$, what is P? What does this say about the relationship between $\vec{y}$, $\vec{p}$, and colsp(X)? *Hint: Use the fact that* $(AB)^{-1} = B^{-1}A^{-1}$.