

# Lab 10: Eigenvalues and Eigenvectors, Convexity

EECS 245, Spring 2026 at the University of Michigan

due for completion at 11:59PM Ann Arbor Time on Monday, June 15th, 2026

Name: \_\_\_\_\_

username: \_\_\_\_\_

Each lab worksheet will contain several activities, some of which will involve writing code and others that will involve writing math on paper. To receive credit for a lab, you must complete as many of the activities as you can in 2 hours and submit a PDF of your work to Gradescope. We will provide specific instructions on how to submit programming activities (e.g. submitting the notebook or including a screenshot of some output).

Feel free to work with others in the course, but you must submit individually.

## Recap: Eigenvalues and Eigenvectors

Let  $A = \begin{bmatrix} 6 & 3 \\ 3 & -2 \end{bmatrix}$ .

- An **eigenvector** of  $A$  is a non-zero vector  $\vec{v}$  such that  $A\vec{v} = \lambda\vec{v}$  for some scalar  $\lambda$ . The scalar  $\lambda$  is called the **eigenvalue** corresponding to  $\vec{v}$ . For  $A$ 's eigenvectors, multiplying by  $A$  is equivalent to multiplying by a scalar.
- The **characteristic polynomial** of  $A$  is given by  $p(\lambda) = \det(A - \lambda I)$ .

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 6 - \lambda & 3 \\ 3 & -2 - \lambda \end{vmatrix} = (6 - \lambda)(-2 - \lambda) - 3 \cdot 3 = \lambda^2 - 4\lambda - 21 = (\lambda + 3)(\lambda - 7)$$

- The eigenvalues of  $A$  are the roots of the characteristic polynomial, so  $\lambda_1 = -3$  and  $\lambda_2 = 7$ .
  - The eigenvector  $\vec{v}_1$  satisfies  $A\vec{v}_1 = -3\vec{v}_1$ .

$$\begin{bmatrix} 6 & 3 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = -3 \begin{bmatrix} a \\ b \end{bmatrix} \implies b = -3a$$

So any vector of the form  $\begin{bmatrix} a \\ -3a \end{bmatrix}$  ( $a \neq 0$ ) is an eigenvector of  $A$  corresponding to the

eigenvalue  $-3$ . We could pick  $\vec{v}_1 = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$ .

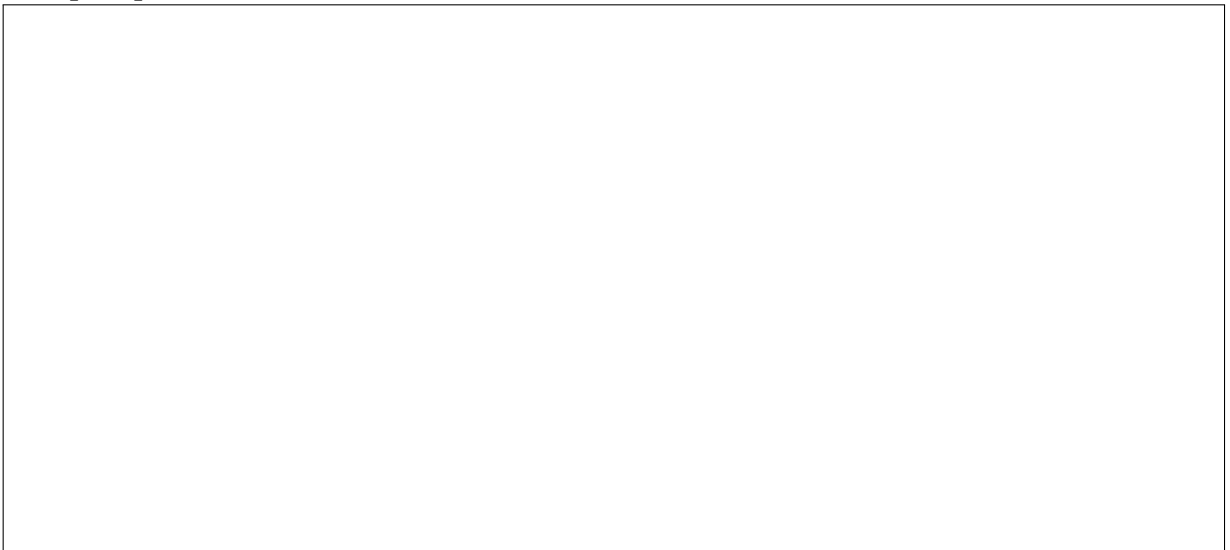
- The eigenvector  $\vec{v}_2$  satisfies  $A\vec{v}_2 = 7\vec{v}_2$ . Another way to find it is to solve for the null space of  $A - 7I = \begin{bmatrix} -1 & 3 \\ 3 & -9 \end{bmatrix}$ . One vector in  $\text{nullsp}(A - 7I)$  is  $\vec{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ .

### Activity 1: Introduction

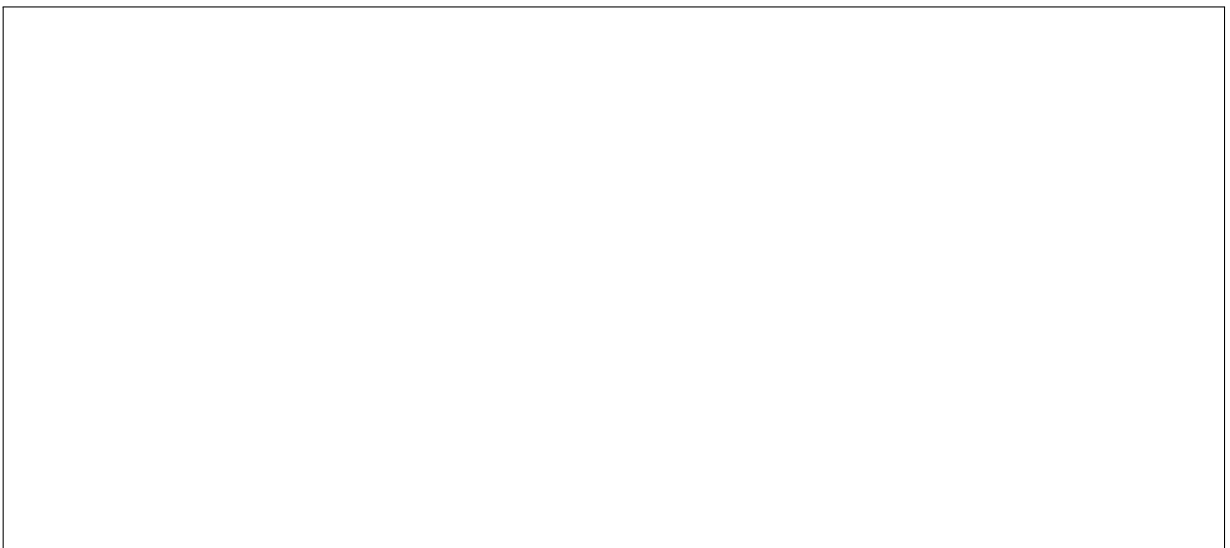
For each  $2 \times 2$  matrix  $A$  below:

- (i) Find the characteristic polynomial of  $A$ , and use it to find the eigenvalues of  $A$ .
- (ii) Find one eigenvector for each eigenvalue of  $A$ . Verify that each eigenvector is indeed an eigenvector of  $A$  by multiplying it by  $A$ .
- (iii) **By hand (not using Python or Desmos)**, draw a picture (like the one in Chapter 9.1 titled [Visualizing the eigenvectors of  \$A\$](#) ) with vectors  $\vec{v}_1, A\vec{v}_1, \vec{v}_2, A\vec{v}_2$  as arrows (where  $\vec{v}_1$  and  $\vec{v}_2$  are the eigenvectors you found above).

a)  $A = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$



b)  $A = \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix}$



## Activity 2: Rapid Fire

The goal of this activity is to practice spotting eigenvalues and characteristic polynomials quickly. Two quick facts:

- The **sum** of the eigenvalues of a matrix is equal to the **trace** of the matrix (which is the sum of the diagonal entries).
- The **product** of the eigenvalues of a matrix is equal to the **determinant** of the matrix.

a) A  $2 \times 2$  matrix  $A$  has  $\text{trace}(A) = 5$  and  $\det(A) = 6$ . What are the eigenvalues of  $A$ ?

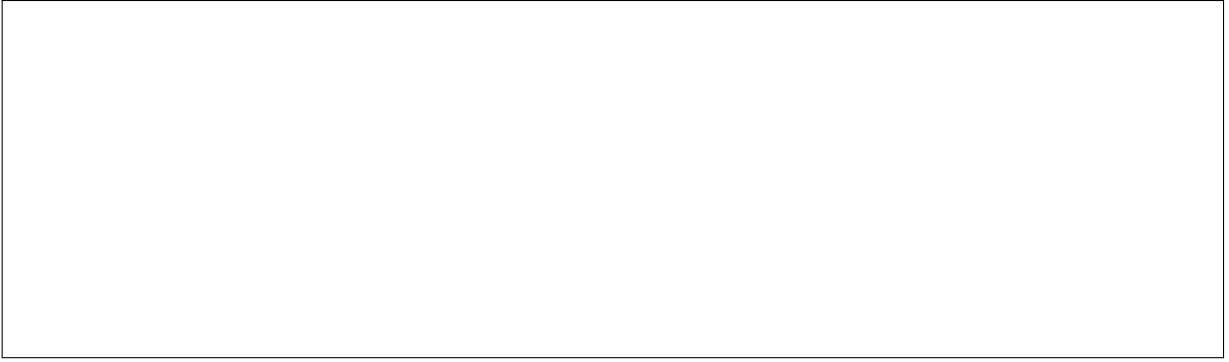
b) A non-invertible  $2 \times 2$  matrix has an eigenvalue of 5. What is its characteristic polynomial?

c) A  $3 \times 3$  matrix  $A$  has  $\det(A) = 20$  and two unique **positive integer** eigenvalues, one of which is repeated twice. In other words,  $p(\lambda)$  has the form

$$p(\lambda) = (\lambda - \lambda_1)^2(\lambda - \lambda_2)$$

( $\lambda_1$  has an **algebraic multiplicity** of 2. This is a term we'll see more in tomorrow's lecture and [Chapter 9.4](#).)

What are all possible values of  $\lambda_1$  and  $\lambda_2$ ?



### Activity 3: Quadratic Forms Return

Open Desmos in 3D mode at [desmos.com/3d](https://www.desmos.com/3d) and write  $z = x^2 + 2bxy + 16y^2$ . This should show you a 3D surface along with a slider for  $b$ . Drag the slider to see how the shape of the surface changes for different  $b$ 's. You should notice that depending on the value of  $b$ , the surface may or may not have a global minimum. Let's explore!

- a)  $z$  is a quadratic form,  $f(\vec{x}) = \vec{x}^T A \vec{x}$ , where  $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$  and  $A$  is a symmetric matrix. Find  $A$ .

- b) For a vector-to-scalar function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ , the **Hessian** of  $f$ , denoted  $\nabla^2 f$ , is the  $n \times n$  matrix of second partial derivatives of  $f$ . Find  $\nabla^2 f$  for  $f(\vec{x}) = \vec{x}^T A \vec{x}$ .

- c) A symmetric matrix  $A$  is **positive semidefinite** (PSD) if  $\vec{v}^T A \vec{v} \geq 0$  for all  $\vec{v} \in \mathbb{R}^n$ . In English, this says that  $A$  is positive semidefinite if the quadratic form  $f(\vec{v}) = \vec{v}^T A \vec{v}$  is always non-negative for all  $\vec{v} \in \mathbb{R}^n$ . Two relevant facts:

- A differentiable vector-to-scalar function  $f$  is **convex** if its Hessian is PSD.
- A symmetric matrix  $A$  is PSD if and only if all of its eigenvalues are non-negative.

Using the facts above, find the range of values  $b$  for which  $f$  is convex, and verify your answer by dragging the slider on Desmos.

#### Activity 4: Understanding Complex Proofs

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a convex function. It turns out that the function  $g(\vec{x})$ , defined by

$$g(\vec{x}) = f(A\vec{x} + \vec{b})$$

for some  $n \times n$  matrix  $A$  and vector  $\vec{b} \in \mathbb{R}^n$ , is also convex, no matter what  $A$  and  $\vec{b}$  are. We're not going to ask you to prove this on your own: instead, we'll give you a proof and ask you questions to ensure you understand it.

---

Our **goal** is to show that  $g((1-t)\vec{x} + t\vec{y}) \leq (1-t)g(\vec{x}) + tg(\vec{y})$ , for all  $\vec{x}, \vec{y} \in \mathbb{R}^n$  and  $t \in [0, 1]$ . We'll start with the "left-hand side" of the definition, and try and leverage  $f$ 's convexity.

$$g((1-t)\vec{x} + t\vec{y}) = f\left(A((1-t)\vec{x} + t\vec{y}) + \vec{b}\right) \tag{1}$$

$$= f\left((1-t)A\vec{x} + tA\vec{y} + \vec{b}\right) \tag{2}$$

$$= f\left((1-t)(A\vec{x} + \vec{b}) + t(A\vec{y} + \vec{b})\right) \tag{3}$$

$$\leq (1-t)f(A\vec{x} + \vec{b}) + tf(A\vec{y} + \vec{b}) \tag{4}$$

$$= \boxed{(1-t)g(\vec{x}) + tg(\vec{y})} \tag{5}$$

---

a) In which line did we use the fact that  $f$  is convex?

b) How did we move from line (1) to line (2), i.e.  $f\left(A((1-t)\vec{x} + t\vec{y}) + \vec{b}\right) = f\left((1-t)A\vec{x} + tA\vec{y} + \vec{b}\right)$ ?

c) How did we move from line (2) to line (3), i.e.  $f\left((1-t)A\vec{x} + tA\vec{y} + \vec{b}\right) = f\left((1-t)(A\vec{x} + \vec{b}) + t(A\vec{y} + \vec{b})\right)$ ?

Recall,  $g(\vec{x}) = f(A\vec{x} + \vec{b})$ , where  $A$  is an  $n \times n$  matrix and  $\vec{x}, \vec{b} \in \mathbb{R}^n$ . On the last page, we showed that if  $f$  is convex, then  $g$  is convex.

Now, let's explore what happens if  $f$  is **strictly** convex. Recall, this means that for all (non-equal)  $\vec{x}$  and  $\vec{y}$  in its domain, and for any  $t \in (0, 1)$ ,

$$f((1-t)\vec{x} + t\vec{y}) < (1-t)f(\vec{x}) + tf(\vec{y})$$

- d) Suppose  $\text{rank}(A) = n$ . Explain why it's impossible for  $A\vec{x} + \vec{b} = A\vec{y} + \vec{b}$  for two different vectors  $\vec{x}$  and  $\vec{y}$ .

- e) Suppose  $\text{rank}(A) < n$ . Explain why it's possible for  $g(\vec{x}) = g(\vec{y})$  for two different vectors  $\vec{x}$  and  $\vec{y}$ . *Hint: Think about  $\text{nullsp}(A)$ .*

- f) Using the above reasoning, explain why if  $f$  is strictly convex, then  $g$  is strictly convex if  $\text{rank}(A) = n$ , and is (not strictly) convex if  $\text{rank}(A) < n$ .

- g) What were your thoughts on this type of activity, where we give you a proof and ask you questions about it?

Hated it    Didn't like it    Neutral    Liked it    Loved it