Lab 11: Adjacency Matrices and Diagonalization Solutions

EECS 245, Spring 2026 at the University of Michigan

due for completion at 11:59PM Ann Arbor Time on Wednesday, June 17th, 2026

Name: ____________________

uniqname: ____________________

Each lab worksheet will contain several activities, some of which will involve writing code and others that will involve writing math on paper. To receive credit for a lab, you must complete as many of the activities as you can in 2 hours and submit a PDF of your work to Gradescope. We will provide specific instructions on how to submit programming activities (e.g. submitting the notebook or including a screenshot of some output).

Feel free to work with others in the course, but you must submit individually.

Recap: Diagonalization (Chapter 9.4)

- Since A has two linearly independent eigenvectors, it is **diagonalizable**, meaning we can write

$$A = V\Lambda V^{-1} = \underbrace{\begin{bmatrix} 2 & 3 \\ -6 & 1 \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} -3 & 0 \\ 0 & 7 \end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix} 0.05 & -0.15 \\ 0.3 & 0.1 \end{bmatrix}}_{V^{-1}}$$

  where V is an invertible matrix whose columns are the eigenvectors of A and Λ is a diagonal matrix with the eigenvalues of A on the diagonal.

  - The **algebraic multiplicity** of an eigenvalue, $\text{AM}(\lambda_i)$, is the number of times it appears as a root of the characteristic polynomial.

$$p(\lambda) = (\lambda - \lambda_1)^{\text{AM}(\lambda_1)}(\lambda - \lambda_2)^{\text{AM}(\lambda_2)} \cdots (\lambda - \lambda_k)^{\text{AM}(\lambda_k)}$$

  - The **geometric multiplicity** of an eigenvalue, $\text{GM}(\lambda_i)$, is the dimension of the eigenspace corresponding to λ_i.

    $\text{GM}(\lambda_i) = \dim(\text{nullsp}(A - \lambda_i I)) = \text{number of linearly independent eigenvectors corresponding to } \lambda_i$

  - For any λ_i, $1 \leq \text{GM}(\lambda_i) \leq \text{AM}(\lambda_i)$.
  - A is diagonalizable if and only if $\text{AM}(\lambda_i) = \text{GM}(\lambda_i)$ for all eigenvalues λ_i. This ensures that A's eigenvectors form a basis of $\mathbb{R}^n$.

- $A = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$ has characteristic polynomial $p(\lambda) = (2-\lambda)^2$. The eigenvalue $\lambda = 2$ has algebraic multiplicity 2, but the corresponding eigenspace is only 1-dimensional:

$$\text{nullsp}(A - 2I) = \text{nullsp}\left(\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$$

  so, $\lambda = 2$ has geometric multiplicity 1. Since $\text{AM}(2) \neq \text{GM}(2)$, A is **not diagonalizable**.

- $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}$ has characteristic polynomial $p(\lambda) = (1-\lambda)^2(3-\lambda)$, so $\lambda_1 = 1$ has $\text{AM}(\lambda_1) = 2$ and $\lambda_2 = 3$ has $\text{AM}(\lambda_2) = 1$. The eigenspace for $\lambda_1 = 1$ is 2-dimensional,

$$\text{nullsp}(A - 1I) = \text{nullsp}\left(\begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right)$$

  so $\text{GM}(\lambda_1) = 2$. Since $\text{AM}(\lambda_i) = \text{GM}(\lambda_i)$ for all eigenvalues λ_i, A is **diagonalizable**.

$$A = V\Lambda V^{-1} = \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix} 0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 \end{bmatrix}}_{V^{-1}}$$

- Diagonalizability is not the same as invertibility!

Activity 1: Rapid Fire

The goal here is to answer the problems **quickly** without working out the details.

a) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$. What are the eigenvalues of A? Is A diagonalizable?

Hint: Use the fact that A is an upper triangular matrix. What is $\det(A - \lambda I)$*?*

Solution: $\lambda_1 = 1, \lambda_2 = 4, \lambda_3 = 6$; A is diagonalizable.

The characteristic polynomial of A is

$$\begin{aligned} p(\lambda) &= \det(A - \lambda I) \\ &= \begin{vmatrix} 1-\lambda & 2 & 3 \\ 0 & 4-\lambda & 5 \\ 0 & 0 & 6-\lambda \end{vmatrix} \\ &= (1-\lambda)\begin{vmatrix} 4-\lambda & 5 \\ 0 & 6-\lambda \end{vmatrix} - 2\begin{vmatrix} 0 & 5 \\ 0 & 6-\lambda \end{vmatrix} + 3\begin{vmatrix} 0 & 4-\lambda \\ 0 & 0 \end{vmatrix} \\ &= (1-\lambda)\left[(4-\lambda)(6-\lambda) - 0\cdot 5\right] - 2\left[0\cdot(6-\lambda) - 0\cdot 5\right] + 3\left[0\cdot 0 - 0\cdot(4-\lambda)\right] \\ &= (1-\lambda)[(4-\lambda)(6-\lambda)] - 0 + 0 \\ &= (1-\lambda)(4-\lambda)(6-\lambda) \end{aligned}$$

The eigenvalues of A are the solutions to $p(\lambda) = 0$, which are $\lambda_1 = 1$, $\lambda_2 = 4$, and $\lambda_3 = 6$. Since A has three distinct eigenvalues, it is diagonalizable.

Why does knowing that all of A's eigenvalues are distinct imply that it's diagonalizable? The key is that for any matrix A, **eigenvectors for different eigenvalues must be linearly independent**. Each eigenvector points in a "new" direction relative to eigenvectors for other eigenvalues. An eigenvector can't correspond to multiple eigenvalues.

From the perspective of multiplicities, remember that the minimum geometric multiplicity of an eigenvalue is 1. Since A has three distinct eigenvalues, they all have geometric multiplicities of 1, which match their algebraic multiplicities.

b) A 5×5 matrix has an eigenvalue of 0 with geometric multiplicity 2. What is its rank?

Solution: $\text{rank}(A) = 3$.

Let A be the matrix in question. Since it has an eigenvalue of 0 with geometric multiplicity 2, it has two linearly independent eigenvectors corresponding to 0. This means that the null space of $A - 0I$, which is the same as the null space of A, has dimension 2.

$$\dim(\text{nullsp}(A)) = 2$$

The rank-nullity theorem tells us that

$$\text{rank}(A) + \dim(\text{nullsp}(A)) = 5$$

Since $\dim(\text{nullsp}(A)) = 2$, we have

$$\text{rank}(A) = 5 - 2 = 3$$

Activity 2: Fundamentals

Let $A = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$.

a) Find the characteristic polynomial of A, and use it to find the eigenvalues of A and their algebraic multiplicities.

Solution: The characteristic polynomial of A is

$$\begin{aligned}
p(\lambda) &= \det(A - \lambda I) \\
&= \begin{vmatrix} 3-\lambda & 2 & 0 \\ 2 & 3-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{vmatrix} \\
&= (3-\lambda)\begin{vmatrix} 3-\lambda & 0 \\ 0 & 5-\lambda \end{vmatrix} - 2\begin{vmatrix} 2 & 0 \\ 0 & 5-\lambda \end{vmatrix} + 0\begin{vmatrix} 2 & 3-\lambda \\ 0 & 0 \end{vmatrix} \\
&= (3-\lambda)\left[(3-\lambda)(5-\lambda) - 0\cdot 0\right] - 2\left[2(5-\lambda) - 0\cdot 0\right] + 0 \\
&= (3-\lambda)\left[(3-\lambda)(5-\lambda)\right] - 2[2(5-\lambda)] \\
&= (3-\lambda)^2(5-\lambda) - 4(5-\lambda) \\
&= (5-\lambda)\left[(3-\lambda)^2 - 4\right] \\
&= (5-\lambda)\left[9 - 6\lambda + \lambda^2 - 4\right] \\
&= (5-\lambda)\left[\lambda^2 - 6\lambda + 5\right] \\
&= (5-\lambda)(\lambda-1)(\lambda-5) \\
&= (1-\lambda)(5-\lambda)^2
\end{aligned}$$

So, A has eigenvalues $\lambda_1 = 1$ with algebraic multiplicity $\text{AM}(\lambda_1) = 1$ and $\lambda_2 = 5$ with algebraic multiplicity $\text{AM}(\lambda_2) = 2$.

b) Find a basis for the eigenspace corresponding to each eigenvalue.

Solution:

- The eigenspace corresponding to $\lambda_1 = 1$ is

$$\text{nullsp}(A - 1I) = \text{nullsp}\left(\begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right)$$

  If you'd rather see this the systems of equations way, we're looking for a vector $\vec{v}_1 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ such that

$$A \begin{bmatrix} a \\ b \\ c \end{bmatrix} = (1) \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$
$$\begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$
$$\begin{bmatrix} 3a + 2b \\ 2a + 3b \\ 5c \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

  The first two equations both are equivalent, and say that $3a + 2b = a \implies 2a + 2b = 0 \implies b = -a$. The last equation just says $5c = c \implies 4c = 0 \implies c = 0$. The easy solution is to let $a = 1$, which gives $\boxed{\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}$. So, the eigenspace is the line of vectors spanned by $\vec{v}_1$.

- The eigenspace corresponding to $\lambda_2 = 5$ is

$$\text{nullsp}(A - 5I) = \text{nullsp}\left(\begin{bmatrix} -2 & 2 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \boxed{\text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)}$$

  If you'd rather see this the systems of equations way, we're looking for a vector $\vec{v}_2 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ such that

$$\begin{bmatrix} 3a + 2b \\ 2a + 3b \\ 5c \end{bmatrix} = \begin{bmatrix} 5a \\ 5b \\ 5c \end{bmatrix}$$

  The first two equations both are equivalent and say that $a = b$. The last equation just says that $c = c$. So, the null space of $A - 5I$ is the set of all vectors with equal first and second components, while the third component can be anything. This is a space spanned by two vectors, $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

c) This particular A is diagonalizable. Diagonalize A by finding a matrix V and a diagonal matrix Λ such that $A = V\Lambda V^{-1}$.

Solution: In the previous part, we saw that $\lambda_1 = 1$ has the eigenvector $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, while $\lambda_2 = 5$ corresponds to the linearly independent eigenvectors $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. So, we can construct V and Λ as follows:

$$V = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

And we have $A = V\Lambda V^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1/2 & -1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$.

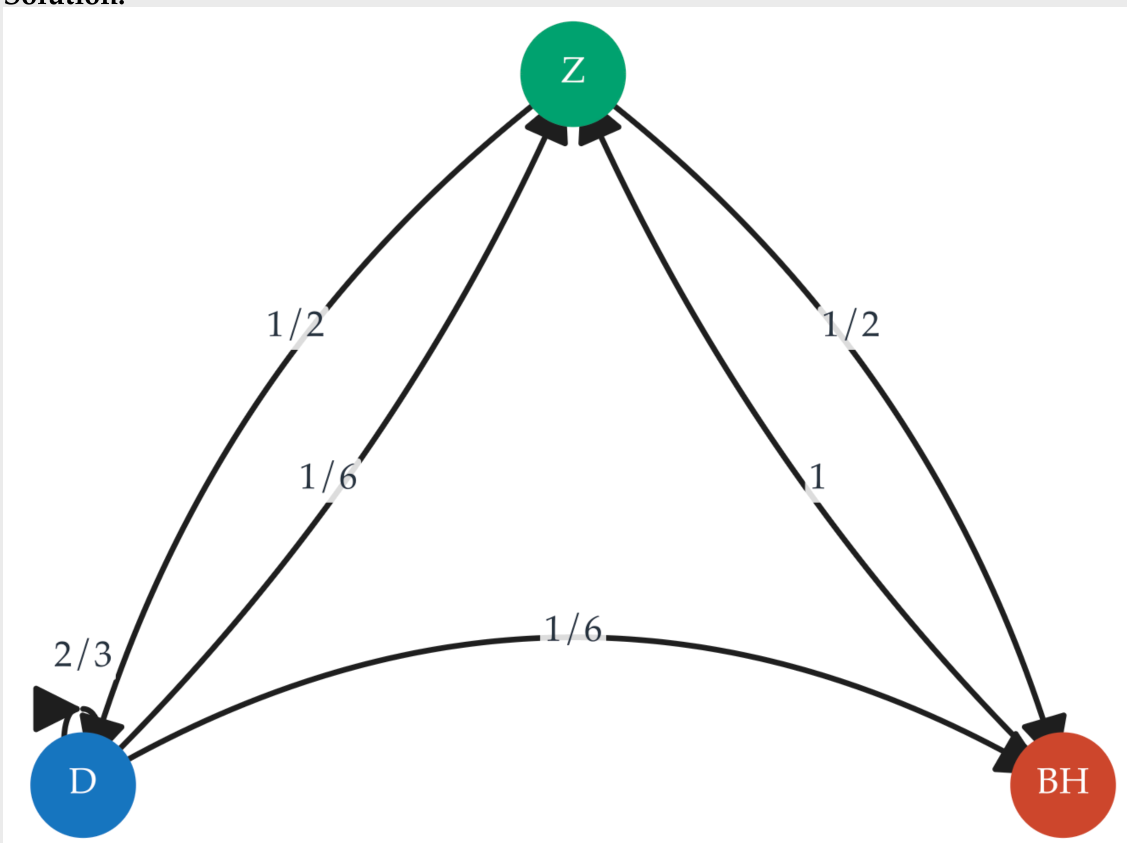
Activity 3: Adjacency Matrices

Suppose that each night, a Wolverine moves between three classic Ann Arbor spots: the Diag, Zingerman's, and the Big House.

- From the Diag, $\frac{2}{3}$ of the time it stays at the Diag, $\frac{1}{6}$ of the time it walks to Zingerman's, and $\frac{1}{6}$ of the time it walks to the Big House.
- From the Big House, it **always** walks to Zingerman's.
- From Zingerman's, it is **equally likely** to walk to the Diag or the Big House.

a) Draw a state diagram for this Markov chain. Make sure to clearly label the edges with their transition probabilities. (For help, see **Chapter 9.3**.)

Solution:



b) Find the adjacency matrix, A, of this Markov chain.

Solution: $A = \begin{bmatrix} p(D \to D) & p(B \to D) & p(Z \to D) \\ p(D \to B) & p(B \to B) & p(Z \to B) \\ p(D \to Z) & p(B \to Z) & p(Z \to Z) \end{bmatrix} = \begin{bmatrix} 2/3 & 0 & 1/2 \\ 1/6 & 0 & 1/2 \\ 1/6 & 1 & 0 \end{bmatrix}$

Let the Diag be state 1, so the first column corresponds to movement **from** the Diag. The second column corresponds to movement from the Big House, and the third column corresponds to movement from Zingerman's.

c) Show that the long-run distribution of the Wolverine's locations is $\begin{bmatrix} p(\text{Diag}) \\ p(\text{Big House}) \\ p(\text{Zingerman's}) \end{bmatrix} = \begin{bmatrix} 6/13 \\ 3/13 \\ 4/13 \end{bmatrix}$.

Hint: Do this by finding the eigenvector of A corresponding to the eigenvalue 1. Since there are infinitely many such eigenvectors, find the one that satisfies the constraint that the components sum to 1.

Solution: We can find the eigenvector corresponding to $\lambda = 1$ by finding a basis for $\text{nullsp}(A - \lambda I)$:

$$\text{nullsp}(A - I) = \text{nullsp}\left(\begin{bmatrix} -1/3 & 0 & 1/2 \\ 1/6 & -1 & 1/2 \\ 1/6 & 1 & -1 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} 6 \\ 3 \\ 4 \end{bmatrix}\right)$$

Dividing $\begin{bmatrix} 6 \\ 3 \\ 4 \end{bmatrix}$ by the sum of its components, 13, gives us the long-run distribution.

d) Open a new Jupyter Notebook or interactive Python session in the Terminal. In it, run:

```
import numpy as np
A = np.array(...) # Replace with the adjacency matrix you found in b)
eigvals, eigvecs = np.linalg.eig(A)
```

Now, if you run `eigvals`, you should see

```
array([ 1.        ,  0.36037961, -0.69371294])
```

If you run `eigvecs[:, 0]` to access the eigenvector corresponding to the first eigenvalue in the array above, you should see

```
array([0.76822128, 0.38411064, 0.51214752])
```

This is **not** the eigenvector you found in the previous part. Why not? What did it return, and what expression can you run in code to find the exact answer you found in the previous part?

> **Solution:** `np.linalg.eig` returns eigenvectors that are **unit vectors**. Since there are infinitely many eigenvectors in any one direction, a unit eigenvector is the one it chooses to return. To find the eigenvector with a sum of 1, run
>
> ```
> eigvecs[:, 0] / np.sum(eigvecs[:, 0])
> ```

e) Run both of the following commands:

```
np.linalg.matrix_power(A, 20) @ np.array([[1/3], [1/3], [1/3]])
np.linalg.matrix_power(A, 21) @ np.array([[1/3], [1/3], [1/3]])
```

What do you see? Why?

> **Solution:** The resulting vectors are the same as the long-run distribution from part **c)** because it's the eigenvector for the eigenvalue 1, the dominant eigenvalue of the adjacency matrix. Let's show why it's converging towards that eigenvector more concretely:
>
> $$\lambda_1 = 1, \lambda_2 = 0.36, \lambda_3 = -0.69, \vec{x} = \begin{bmatrix} 1/3 & 1/3 & 1/3 \end{bmatrix}^T$$
>
> All of A's eigenvectors, $\vec{v}_1, \vec{v}_2$, and $\vec{v}_3$ are linearly independent, so $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$ for some constants c_1, c_2, c_3.
>
> $$\begin{aligned} A^k\vec{x} &= A^k(c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3) \\ &= c_1A^k\vec{v}_1 + c_2A^k\vec{v}_2 + c_3A^k\vec{v}_3 \\ &= c_1\lambda_1^k\vec{v}_1 + c_2\lambda_2^k\vec{v}_2 + c_3\lambda_3^k\vec{v}_3 \\ &= c_1\lambda_1^k\vec{v}_1 \end{aligned}$$
>
> As $k \to \infty$, λ_2^k and λ_3^k will both approach 0 because their magnitudes are less than 1, while λ_1^k stays at 1. This leaves us with the eigenvector for λ_1.

f) Now, consider the adjacency matrix

$$B = \begin{bmatrix} 0 & 0 & 1/2 \\ 0 & 0 & 1/2 \\ 1 & 1 & 0 \end{bmatrix}$$

Using `numpy`, find the eigenvalues of B. Then, run all of the following commands:

```
np.linalg.matrix_power(B, 20) @ np.array([[1/3], [1/3], [1/3]])
np.linalg.matrix_power(B, 21) @ np.array([[1/3], [1/3], [1/3]])
np.linalg.matrix_power(B, 22) @ np.array([[1/3], [1/3], [1/3]])
np.linalg.matrix_power(B, 23) @ np.array([[1/3], [1/3], [1/3]])
```

This Markov chain appears not to converge. **Why not?** Relate your answer to the discussion of "the dominant eigenvalue" in **Chapter 9.3**.

Solution: The eigenvalues of B are $\lambda_1 = -1, \lambda_2 = 0, \lambda_3 = 1$. Let's take a closer look at what happens when we apply the same idea from part **e)**:

$$\begin{aligned} A^k\vec{x} &= A^k(c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3) \\ &= c_1A^k\vec{v}_1 + c_2A^k\vec{v}_2 + c_3A^k\vec{v}_3 \\ &= c_1\lambda_1^k\vec{v}_1 + c_3\lambda_3^k\vec{v}_3 \\ \\ \text{if } k \text{ is even, } &= c_1\lambda_1^k\vec{v}_1 + c_3\lambda_3^k\vec{v}_3 \\ \text{if } k \text{ is odd, } &= c_1\lambda_1^k\vec{v}_1 - c_3\lambda_3^k\vec{v}_3 \end{aligned}$$

There are **2** eigenvalues with magnitude 1, and one of them is negative. As $k \to \infty$, the result of $A^k\vec{x}$ will bounce between two different vectors.

Activity 4: Symmetric Matrices

Suppose A is a symmetric $n \times n$ matrix, meaning $A = A^T$. Symmetric matrices have a special property, described by the spectral theorem: **the eigenvectors corresponding to different eigenvalues are orthogonal**. Below, we provide a proof of this fact.

1. Let $\vec{v}_i$ be an eigenvector of A with eigenvalue λ_i, so $A\vec{v}_i = \lambda_i\vec{v}_i$.
2. Let $\vec{v}_j$ be an eigenvector of A with eigenvalue λ_j, where $\lambda_i \neq \lambda_j$, so $A\vec{v}_j = \lambda_j\vec{v}_j$.
3. The dot product of $\vec{v}_i$ and $A\vec{v}_j$ is $\vec{v}_i \cdot (A\vec{v}_j) = \vec{v}_i \cdot (\lambda_j\vec{v}_j) = \lambda_j(\vec{v}_i \cdot \vec{v}_j)$.
4. But also, $\vec{v}_i \cdot (A\vec{v}_j) = \vec{v}_i^T A\vec{v}_j = \vec{v}_i^T A^T\vec{v}_j = (A\vec{v}_i)^T\vec{v}_j = \lambda_i\vec{v}_i^T\vec{v}_j = \lambda_i(\vec{v}_i \cdot \vec{v}_j)$.
5. The final expressions in both cases are equal, so $\lambda_j(\vec{v}_i \cdot \vec{v}_j) = \lambda_i(\vec{v}_i \cdot \vec{v}_j)$.
6. Equivalently, $(\lambda_j - \lambda_i)(\vec{v}_i \cdot \vec{v}_j) = 0$. But since $\lambda_i \neq \lambda_j$, we must have $\vec{v}_i \cdot \vec{v}_j = 0$.

a) In which line did we use the fact that A is symmetric? Select the line below, and then in that line, circle the specific part of the equation that uses the fact that $A = A^T$.

○ 1 ○ 2 ○ 3 ● 4 ○ 5 ○ 6

Solution: Line 4 is the only line that uses the fact that $A = A^T$, but it's the most important line in the proof. Specifically, here's where we used that fact:

$$\vec{v}_i^T A\vec{v}_j = \vec{v}_i^T A^T\vec{v}_j$$

By using $A = A^T$ here, we were able to write $\vec{v}_i \cdot (A\vec{v}_j)$ as $\vec{v}_i^T A^T\vec{v}_j = (A\vec{v}_i) \cdot \vec{v}_j$, which was key to our final expression.

b) The **spectral theorem** states that any symmetric $n \times n$ matrix A can be diagonalized by an orthogonal matrix Q through

$$A = Q\Lambda Q^T$$

Explain how this relates to the "regular" diagonalization $A = V\Lambda V^{-1}$. Why does the equation above contain Q^T instead of Q^{-1}?

Solution: A standard (not necessarily symmetric) matrix A that is diagonalizable can be written as

$$A = V\Lambda V^{-1}$$

If A is symmetric and can be written $A = Q\Lambda Q^T$, then the fact that we have Q^T instead of Q^{-1} stems from the fact that orthogonal matrices satisfy $Q^T = Q^{-1}$. This comes from the fact that $Q^TQ = QQ^T = I$. Transposes are easier to compute than inverses, and easier to interpret, too: if Q is a rotation, then Q^T is the inverse rotation, i.e. a rotation by the opposite amount.

c) Symmetric matrices play a big role in solving optimization problems in machine learning. You'll get a taste of this in Homework 10, Problem 6, which discusses *ridge regression* and *regularization*, ideas that we use to make sure that our models are not overfitting to training data.

Here, we'll prove a related fact: **if A is an $n \times n$ symmetric matrix and all of its eigenvalues are non-negative, then A is positive semidefinite.** A symmetric matrix A is positive semidefinite if and only if $\boxed{\vec{v}^T A \vec{v} \geq 0}$ for all $\vec{v} \in \mathbb{R}^n$. One interpretation: this means the quadratic form $f(\vec{v}) = \vec{v}^T A \vec{v}$ is always greater than or equal to 0, no matter what $\vec{v}$ is.

Let's work through the proof step-by-step: we will do some of it for you, and you'll complete the rest. (To be precise, we're only proving one direction of the "if and only if" statement; the other direction is in Homework 10!) First, since A is symmetric, the **spectral theorem** tells us that A can be written as

$$A = Q\Lambda Q^T$$

where Q is an orthogonal matrix and Λ is a diagonal matrix with the eigenvalues of A on the diagonal. This is the eigenvector decomposition of A.

On top of that, suppose that all of A's eigenvalues, $\lambda_1, \lambda_2, \ldots, \lambda_n$, are non-negative.

Now, let $\vec{v}$ be some arbitrary vector (not necessarily an eigenvector of A) in $\mathbb{R}^n$. Then, eventually we need to show that $\vec{v}^T A \vec{v} \geq 0$, regardless of what $\vec{v}$ is. Let's start by expanding $\vec{v}^T A \vec{v}$ using the fact that $A = Q\Lambda Q^T$.

$$\vec{v}^T A \vec{v} = \vec{v}^T (Q\Lambda Q^T)\vec{v} = (\vec{v}^T Q)\Lambda(Q^T \vec{v})$$

Suppose that $\vec{y} = Q^T \vec{v}$. Then, $\vec{y}^T = (Q^T \vec{v})^T = \vec{v}^T Q$. This seems like an arbitrary maneuver, but it will be useful in a moment.

$$\vec{v}^T A \vec{v} = \vec{y}^T \Lambda \vec{y} = \begin{bmatrix} y_1 & y_2 & \ldots & y_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

Your job is to complete the rest of the proof. Show that if A is an $n \times n$ symmetric matrix and all of its eigenvalues are non-negative, then $\vec{v}^T A \vec{v} \geq 0$ for all $\vec{v} \in \mathbb{R}^n$.

Solution:

$$\begin{aligned}
\vec{v}^T A \vec{v} &= \vec{y}^T \Lambda \vec{y} \\
&= \begin{bmatrix} y_1 & y_2 & \dots & y_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \\
&= \begin{bmatrix} y_1 & y_2 & \dots & y_n \end{bmatrix} \begin{bmatrix} \lambda_1 y_1 \\ \lambda_2 y_2 \\ \vdots \\ \lambda_n y_n \end{bmatrix} \\
&= \sum_{i=1}^{n} \lambda_i y_i^2
\end{aligned}$$

Remember that none of the λ_i's are negative. So, each term $\lambda_i y_i^2$ is non-negative too, meaning the entire sum $\sum_{i=1}^{n} \lambda_i y_i^2$ is non-negative. What we've shown is that if $\vec{v}$ is **any** vector in $\mathbb{R}^n$, then $\vec{v}^T A \vec{v}$ ends up being a sum of this form, which is always non-negative, so $\vec{v}^T A \vec{v} \geq 0$, and thus A is positive semidefinite.

Activity 5: More Practice (Optional)

Let A be a 3×3 with:

- eigenvalue $\lambda_1 = 3$ with eigenvector $\vec{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.
- eigenvalue $\lambda_2 = -2$ with the 2-dimensional eigenspace: span $\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right)$.

a) Find A. Feel free to use `numpy` to find an inverse for you and verify your answer.

Solution: $A = \begin{bmatrix} 0.5 & -2.5 & 5 \\ 0 & -2 & 0 \\ 1.25 & -1.25 & 0.5 \end{bmatrix}$.

The eigenvector decomposition of A is $A = V\Lambda V^{-1}$, where V is the matrix whose columns are the eigenvectors of A and Λ is the diagonal matrix with the eigenvalues of A on the diagonal, arranged in the same order as the eigenvectors in V. We know that $\lambda_1 = 3$ with eigenvector $\vec{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ and that $\lambda_2 = \lambda_3 = -2$ with eigenvectors $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$.

Placing this information into V and Λ gives us

$$V = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

So,

$$A = V\Lambda V^{-1} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \left(\begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} \right)^{-1}$$

A bit of help from `numpy` shows that $A = \begin{bmatrix} 0.5 & -2.5 & 5 \\ 0 & -2 & 0 \\ 1.25 & -1.25 & 0.5 \end{bmatrix}$.

```
>>> V = np.array([
    [2, 1, 0],
    [0, 1, 2],
    [1, 0, 1]
])
>>> V @ np.diag([3, -2, -2]) @ np.linalg.inv(V)
array([[ 0.5 , -2.5 ,  5.  ],
       [ 0.  , -2.  ,  0.  ],
       [ 1.25, -1.25,  0.5 ]])
```

b) Let V be the matrix whose columns are the eigenvectors of A and Λ be the diagonal matrix with the eigenvalues of A on the diagonal. In terms of V and Λ, what is A^8?

Solution: $A^8 = V\Lambda^8 V^{-1}$.
This comes from the fact that $A = V\Lambda V^{-1}$, so

$$A^8 = V\Lambda(V^{-1}V)\Lambda(V^{-1}V)\cdots\Lambda V^{-1} = V\Lambda^8 V^{-1}$$

The sequential multiplication of V^{-1} with V cancels out, and we stack together 8 copies of Λ.
Since Λ is a diagonal matrix, we can raise each diagonal element to the 8th power.

$$\Lambda^8 = \begin{bmatrix} 3^8 & 0 & 0 \\ 0 & (-2)^8 & 0 \\ 0 & 0 & (-2)^8 \end{bmatrix}$$

So, $A^8 = V \begin{bmatrix} 3^8 & 0 & 0 \\ 0 & (-2)^8 & 0 \\ 0 & 0 & (-2)^8 \end{bmatrix} V^{-1}$.

c) Suppose $\vec{x} \in \mathbb{R}^3$ is some vector. As $k \to \infty$, what does $A^k\vec{x}$ approach? Explain your answer in English.

Solution: As $k \to \infty$, $A^k\vec{x}$ approaches the eigenvector corresponding to $\lambda_1 = 3$ (the dominant eigenvalue), so some scalar multiple of $\vec{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.

Remember that $A^k = V\Lambda^k V^{-1}$. As k increases, the diagonal entries of Λ^k — $3^k, (-2)^k, (-2)^k$ — all grow exponentially larger, but 3^k grows faster than $(-2)^k$. So, the contribution of the second and third eigenvectors to $A^k\vec{x}$ diminishes, and $A^k\vec{x}$ approaches a scalar multiple of $\vec{v}_1$.

To be clear, the numbers in $A^k\vec{x}$ will approach infinity; it's the **direction** of $A^k\vec{x}$ that approaches the direction of $\vec{v}_1$. Simulate this in numpy to see for yourself!

```
>>> x = np.array([[1], [1], [1]])
>>> A = np.array([
    [0.5, -2.5, 5],
    [0, -2, 0],
    [1.25, -1.25, 0.5]
])
>>> x_k = np.linalg.matrix_power(A, 50) @ x
>>> x_k # Massive numbers!
array([[7.17897988e+23],
       [1.12589991e+15],
       [3.58948994e+23]])
>>> x_k / np.linalg.norm(x_k) # Unit vector makes numbers smaller.
array([[8.94427191e-01],
       [1.40275570e-09],
       [4.47213596e-01]])
# Notice that the unit vector is roughly [2, 0, 1] multiplied by a scalar.
```