EECS 245, Spring 2026

LEC 3 Vectors and the Dot Product

→ Read: Ch. 3.1 - 3.3

Agenda

keep up with readings!

- Brief recap: simple linear regression } 2.1-2.4
- What do we need vectors for? } 2.5
- Vectors
  - Definition
  - Norm
  - Addition
  - Scalar multiplication

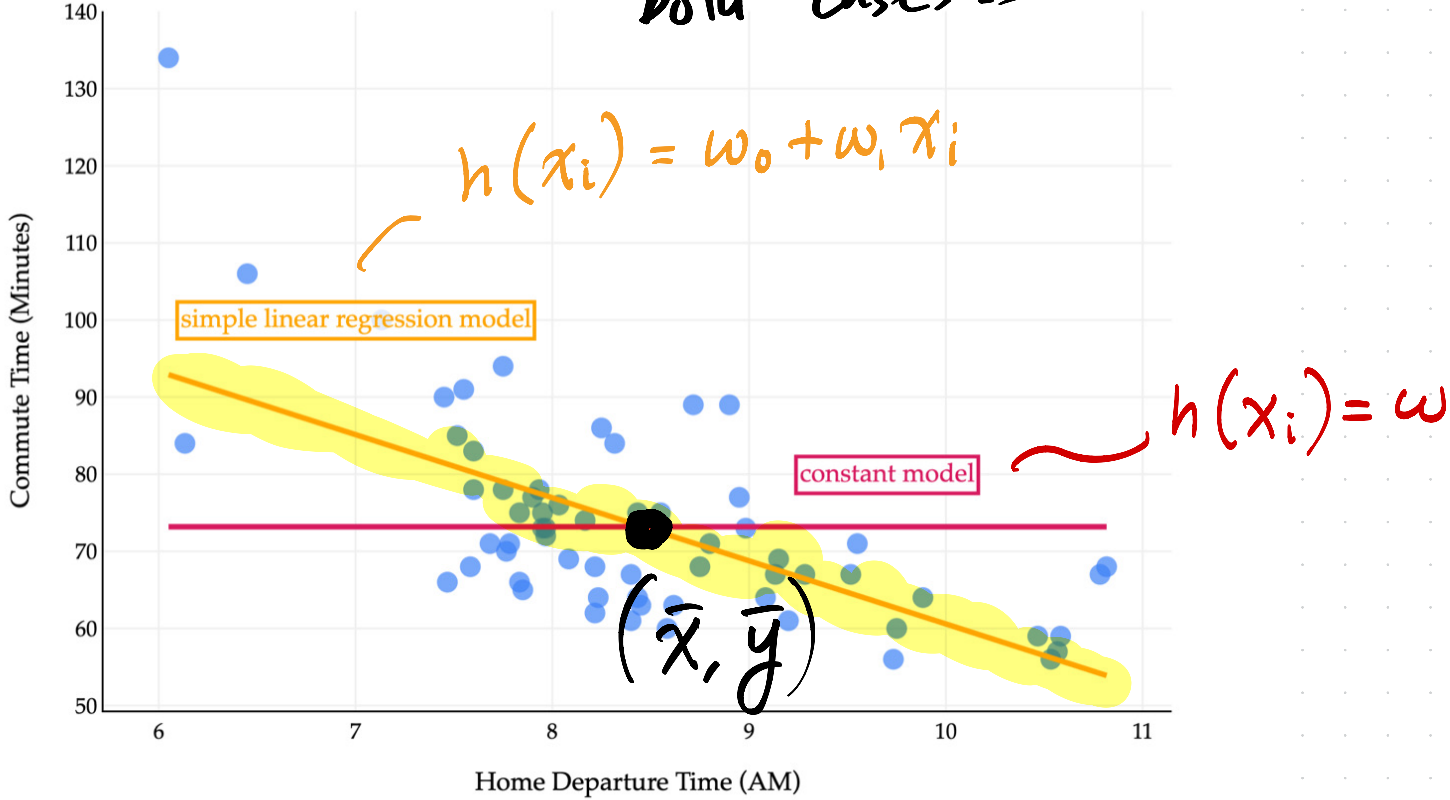
  } 3.1-3.2
- The dot product } 3.3

Announcements

- HW 2 due tomorrow
- Lab 3 due tomorrow
- Solutions for Lab 1, Lab 2, HW 1 all up

  needed for HW 1 → HW 1
- Midterm 1 next Friday!

  → one handwritten double-sided 8.5 x 11" notes sheet suppose we use squared loss in both cases ....



Three-step modeling process

① Choose a model

- $h(x_i) = w$
- $h(x_i) = w_0 + w_1 x_i$

② Choose a loss function

- squared loss

$$L_{sq}(y_i, h(x_i)) = (y_i - h(x_i))^2$$

- absolute loss

③ Minimize average loss to find optimal parameters for constant model, absolute loss:

$$R_{abs}(w) = \frac{1}{n}\sum_{i=1}^{n} |y_i - w|$$

for simple linear regression model, squared loss:

$$R_{sq}(w_0, w_1) = \frac{1}{n}\sum_{i=1}^{n} \left(y_i - (w_0 + w_1 x_i)\right)^2$$

$$\Rightarrow w_1^* = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sum (x_i - \bar{x})^2} = r\frac{\sigma_y}{\sigma_x}$$

best slope

correlation

$$\Rightarrow w_0^* = \bar{y} - w_1^* \bar{x}$$

	date	day	departure_hour	minutes
0	5/15/2023	Mon	10.816667	68.0
1	5/16/2023	Tue	7.750000	94.0
2	5/22/2023	Mon	8.450000	63.0
3	5/23/2023	Tue	7.133333	100.0
4	5/30/2023	Tue	9.150000	69.0

multiple linear regression: use multiple features!

$$h(\text{dept. hour}_i, \text{dom}_i) = w_0 + w_1 \overbrace{(\text{dept. hour}_i)}^{x_i^{(1)}} + w_2 \underbrace{(\text{dom}_i)}_{x_i^{(2)}}$$

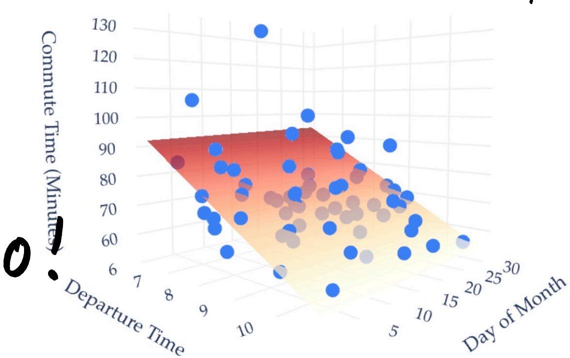
$$h(\text{dept. hour}_i, \text{dom}_i) = w_0 + w_1 \overbrace{(\text{dept. hour}_i)} + w_2 \underbrace{(\text{dom}_i)}_{x_i^{(2)}}$$

How would we find w_0^*, w_1^*, w_2^*?

$$R_{sq}(w_0, w_1, w_2) = \frac{1}{n} \sum_{i=1}^{n} \left(y_i - \left(w_0 + w_1 (dh)_i + w_2 (dom)_i \right) \right)^2$$

$\Rightarrow$ take $\dfrac{\partial R}{\partial w_0}, \dfrac{\partial R}{\partial w_1}, \dfrac{\partial R}{\partial w_2}$

and solve for where all are 0!



Vectors (Ch. 3)

a **vector** is an ordered list of numbers

$$\vec{v} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} \qquad \vec{w} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

$\vec{v} \neq \vec{w}$

3 components/elements

in

$$\vec{v} \in \mathbb{R}^3$$

"$\vec{v}$ is in R three"

set of all vectors with 3 components, all of which are real numbers

General: $\vec{v} \in \mathbb{R}^n$

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

here, each v_i is a scalar (single number)

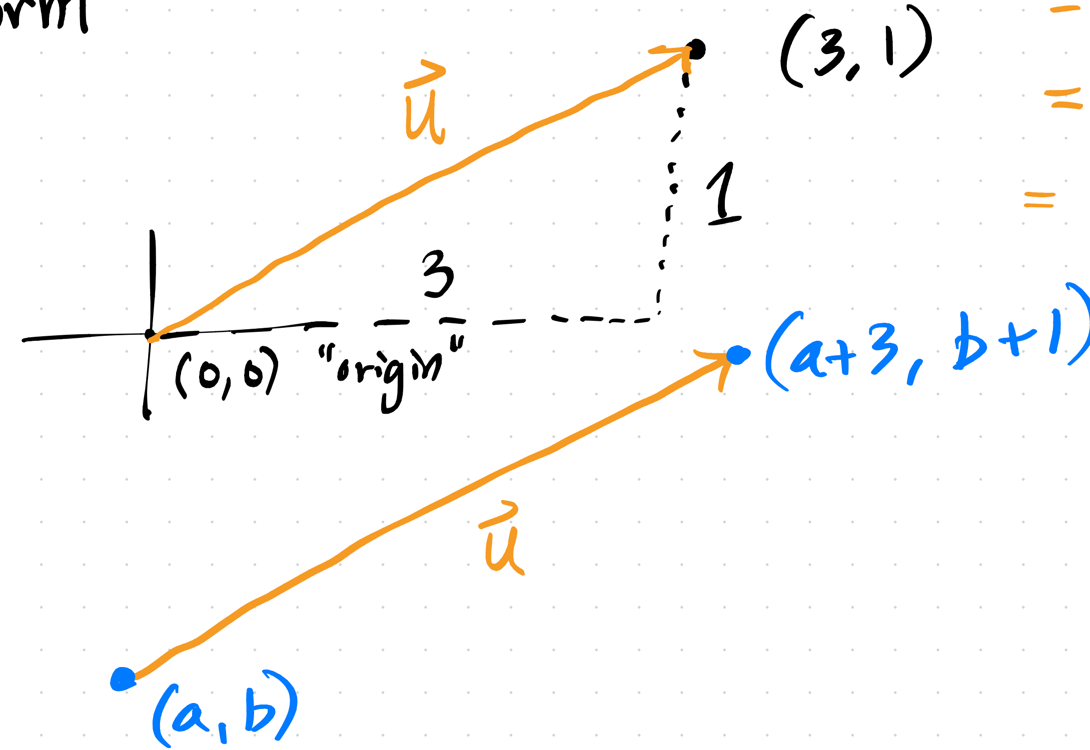
in another context, $\vec{v}_i$ could also be a vector

Length

also called its norm

$$\vec{u} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\|\vec{u}\| = \text{length} = \sqrt{3^2 + 1^2} = \sqrt{10}$$



$\vec{v} \in \mathbb{R}^n$

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} = \sqrt{\sum_{i=1}^{n} v_i^2}$$

L2 norm which is the default (other norms exist too, like L1)

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$\|\vec{v}\|$ "length" "norm" "magnitude"

Ch. 3.2

2 important operations that vectors support:

1. addition
2. scalar multiplication

Addition

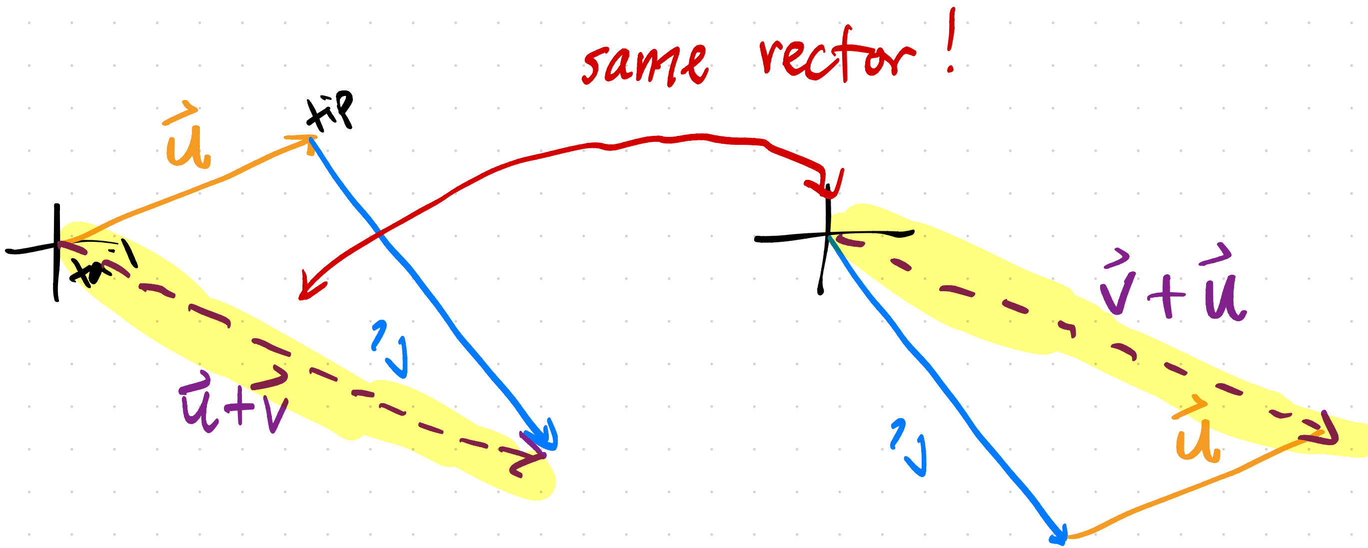
$$\underbrace{\vec{u},\ \vec{v} \in \mathbb{R}^n}_{\text{both have the same \# of components}}$$

"element-wise"

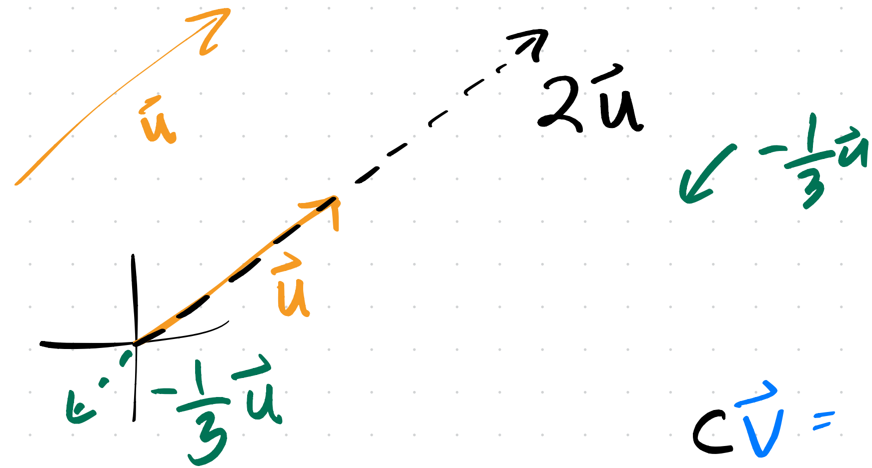
$$\vec{u} + \vec{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix} = \vec{v} + \vec{u}$$

vector addition is "commutative"

$$\vec{u} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} 4 \\ -6 \end{bmatrix} \qquad \vec{u}+\vec{v} = \begin{bmatrix} 7 \\ -5 \end{bmatrix}$$



② scalar multiplication → multiplying a vector by a single number



$$c\vec{v} = \begin{bmatrix} c\,v_1 \\ c\,v_2 \\ \vdots \\ c\,v_n \end{bmatrix}$$

all scalar multiples of $\vec{u}$ live on the same line, i.e. are parallel!

all that changes is length (and orientation)

"Linear combination"

"a little bit of $\vec{u}$ plus a little bit of $\vec{v}$"

$$3\vec{u} - \frac{1}{2}\vec{v}$$

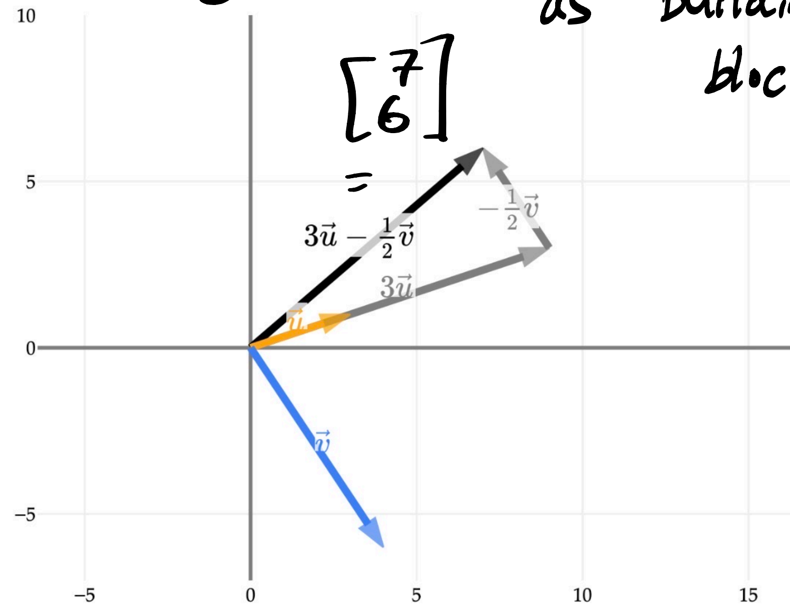
$$= 3\begin{bmatrix} 3 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 4 \\ -6 \end{bmatrix}$$

$$= \begin{bmatrix} 7 \\ 6 \end{bmatrix}$$

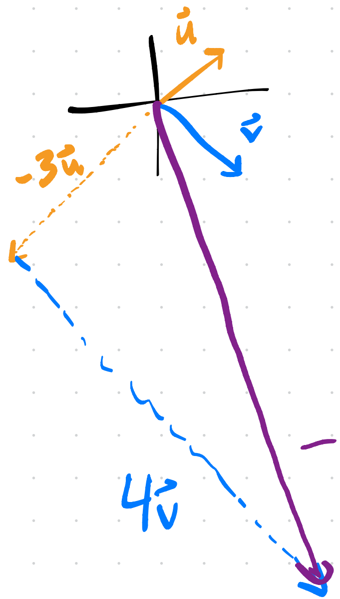
$$\vec{u} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} 4 \\ -6 \end{bmatrix}$$

key idea: think of $\vec{u}, \vec{v}$ as building blocks!

$$\begin{bmatrix} 7 \\ 6 \end{bmatrix} =$$



$\vec{u} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 4 \\ -6 \end{bmatrix}$



$-3\vec{u} + 4\vec{v} = \begin{bmatrix} 7 \\ -27 \end{bmatrix}$

General definition :

suppose we have d vectors,

$\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d$, all of which are in $\mathbb{R}^n$

A linear combination of these vectors is any other vector that can be written as

$$a_1 \vec{v}_1 + a_2 \vec{v}_2 + a_3 \vec{v}_3 + \cdots + a_d \vec{v}_d$$

where $a_1, a_2, \ldots, a_d$ are scalars! (coefficients)

$$3\vec{v}_1 + 4\vec{v}_2 - 17\vec{v}_{54} + 12\vec{v}_{91} + \cdots$$

Activity

building blocks

$$\vec{x} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \quad \vec{y} = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$$

Goal: Write

$$\vec{z} = \begin{bmatrix} 9 \\ -16 \\ -1 \end{bmatrix}$$

as a linear combination of $\vec{x}$ and $\vec{y}$

$$a\vec{x} + b\vec{y} = \vec{z}$$

$$a\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} + b\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 9 \\ -16 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 3a + b \\ -a + 4b \\ 2a + 3b \end{bmatrix} = \begin{bmatrix} 9 \\ -16 \\ -1 \end{bmatrix}$$

2:16 PM Tue May 12

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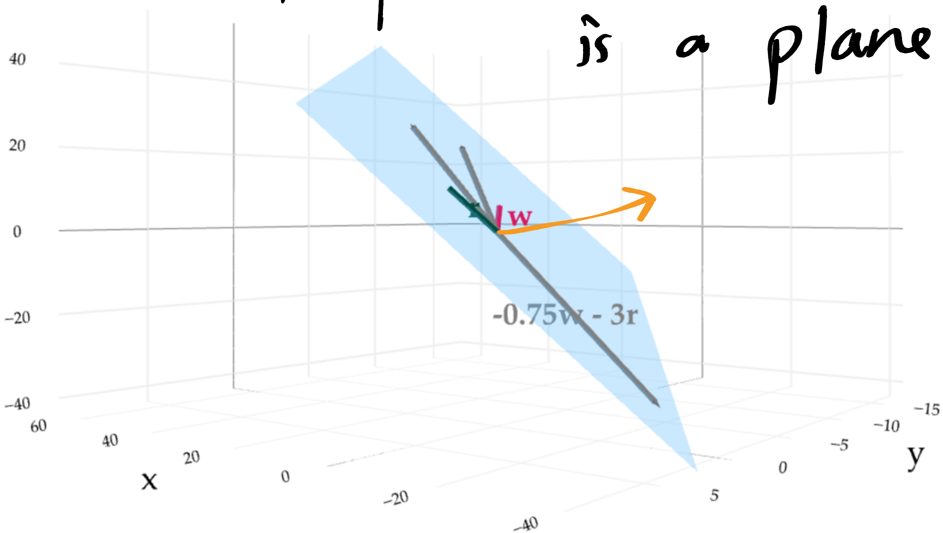
Toggle Plane

The set of all linear combinations of 2 non-parallel vectors in $\mathbb{R}^3$ is a plane!



$$\begin{bmatrix} 3a+b \\ -a+4b \\ 2a+3b \end{bmatrix} = \begin{bmatrix} 9 \\ -16 \\ -1 \end{bmatrix} \begin{matrix} ① \\ ② \\ ③ \end{matrix}$$

Solve for a, b!

② : $a = 16 + 4b$

into ① $3(16+4b) + b = 9$

$48 + 12b + b = 9$

$13b = -39$

$b = -3$

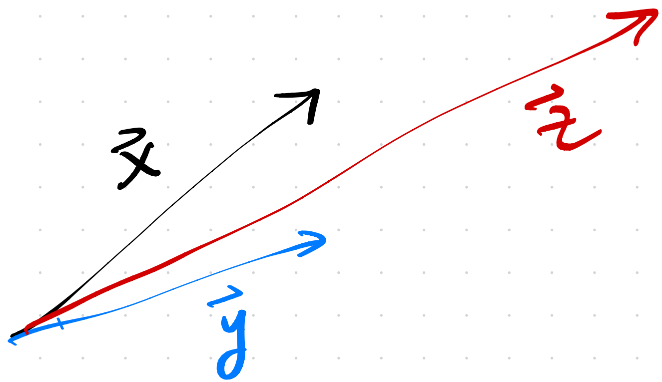
$a = 16 + 4b = 16 + 4(-3) = 4$

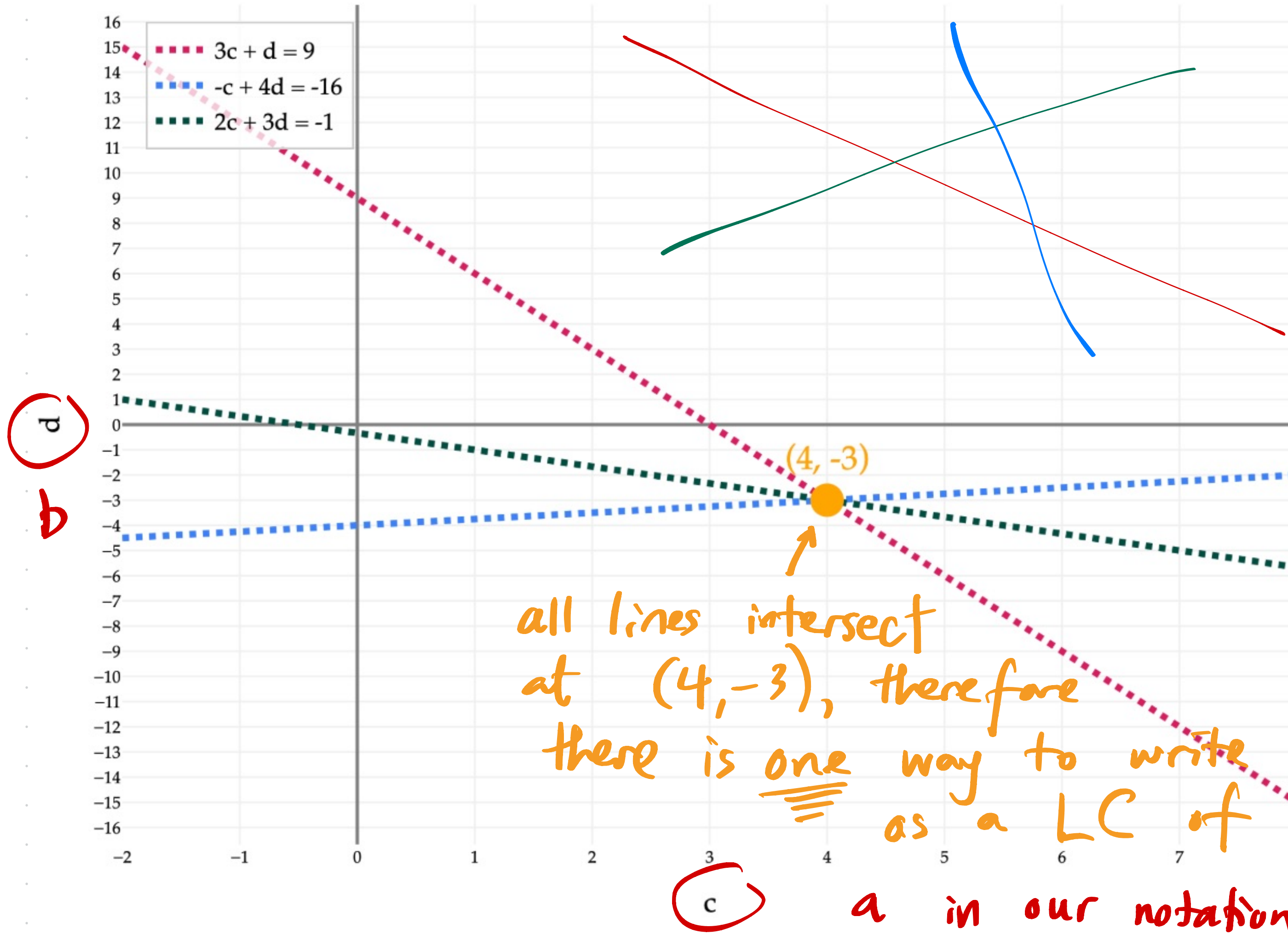
So, $a = 4$, $b = -3$ works!

$(4, -3)$ satisfies all three equations!

$$\vec{x} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \quad \vec{y} = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$$

$$\Rightarrow \quad 4\vec{x} - 3\vec{y} = \begin{bmatrix} 9 \\ -16 \\ -1 \end{bmatrix}$$





all lines intersect at (4, -3), therefore there is **one** way to write $\vec{z}$ as a LC of $\vec{x}, \vec{y}$.

b

a in our notation

Ch. 3.3: Dot Product

$$\underbrace{\vec{u}, \vec{v} \in \mathbb{R}^n}_{\text{same \# of components}}$$

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

commutative!

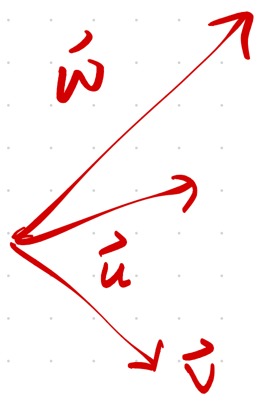
$$\vec{u} \cdot \vec{v} = \underbrace{u_1 v_1 + u_2 v_2 + \cdots + u_n v_n}_{\text{a scalar! not a vector}} = \vec{v} \cdot \vec{u}$$

$$\vec{x} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \qquad \vec{y} = \begin{bmatrix} -2 \\ -5 \end{bmatrix} \qquad \vec{x} \cdot \vec{y} = 4(-2) + 3(-5) = -8 - 15 = -23$$

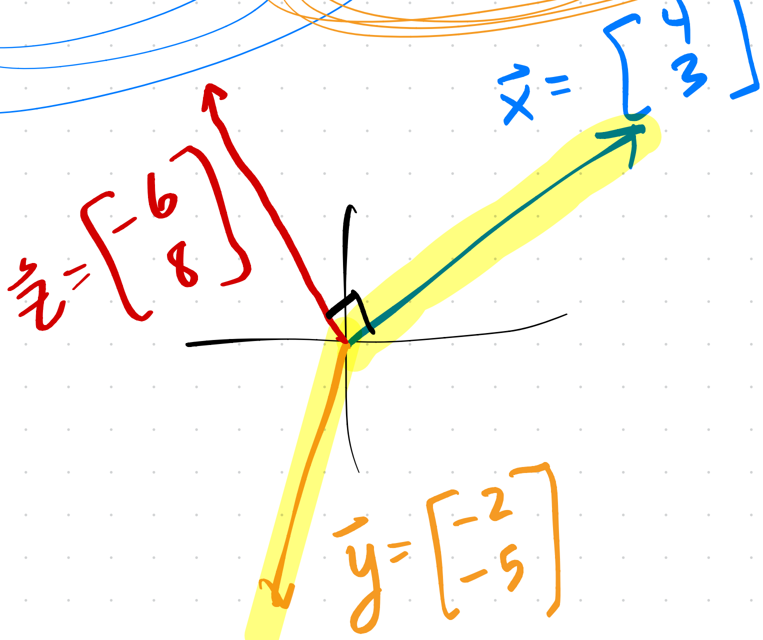
$\cancel{\vec{u} \cdot \vec{v} \cdot \vec{w}}$ not defined!

$$(\vec{u} \cdot \vec{v})\,\vec{w} \neq \vec{u}\,(\vec{v} \cdot \vec{w})$$

$$(\vec{v} \cdot \vec{w})\,\vec{u}$$



$\vec{x} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ $\vec{y} = \begin{bmatrix} -2 \\ -5 \end{bmatrix}$



$\vec{x} \cdot \vec{y} = -23$

$\vec{x} \cdot \vec{z} = 4(-6) + 3(8) = 0$

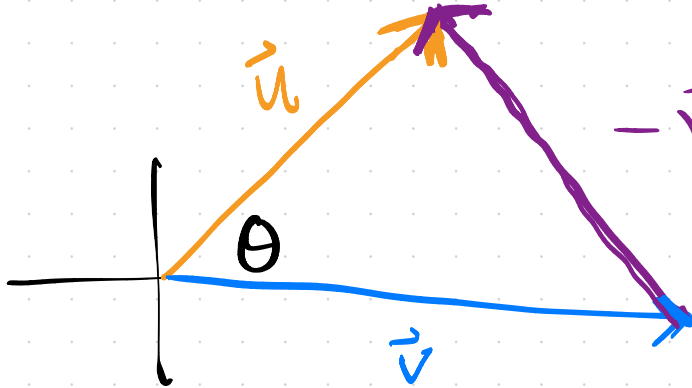
dot product of 0 $\Leftrightarrow$ "perpendicular" "orthogonal"

$\vec{x} \cdot \vec{x} = (4)(4) + (3)(3) = 25 = \|\vec{x}\|^2$

Key idea: dot product measures

similarity

but why? what is the connection between the dot product and angles???

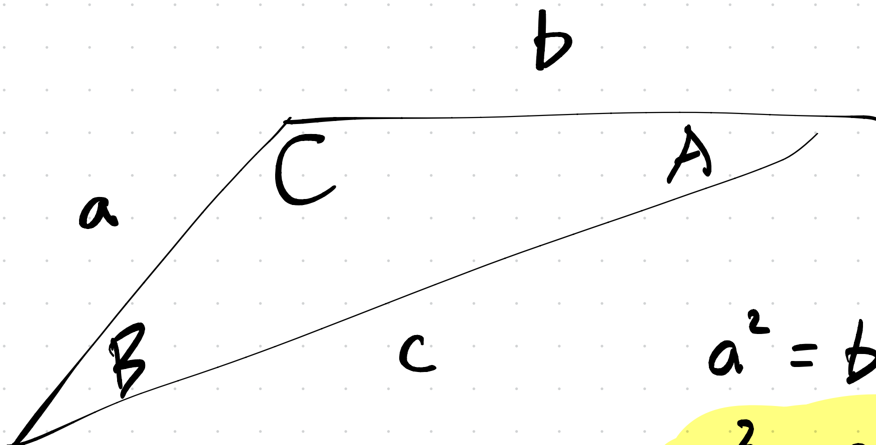


$$-\vec{v} + \vec{u} = \vec{u} - \vec{v}$$

In general: $c^2 = a^2 + b^2 - 2ab\cos(C)$

Here: $\|\vec{u} - \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\|\cos\theta$

Aside: cosine law



$$a^2 = b^2 + c^2 - 2bc\cos(A)$$

$$c^2 = a^2 + b^2 - 2ab\cos(C)$$

We've discovered :

$$\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\|\cos\theta$$

$$\|\vec{u}-\vec{v}\|^2 = (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v})$$

$$= \underline{\vec{u}\cdot\vec{u}} - \underline{\vec{u}\cdot\vec{v}} - \underline{\vec{v}\cdot\vec{u}} + \underline{\vec{v}\cdot\vec{v}}$$

$$= \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\,\vec{u}\cdot\vec{v}$$

$$\cancel{\|\vec{u}\|^2} + \cancel{\|\vec{v}\|^2} \underset{-2}{\underline{-2}} \|\vec{v}\|\|\vec{v}\|\cos\theta = \cancel{\|\vec{u}\|^2} + \cancel{\|\vec{v}\|^2} \underset{-2}{\underline{-2}}\,\vec{u}\cdot\vec{v}$$

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta$$

"geometric definition"

$$= u_1 v_1 + u_2 v_2 + \dots + u_n v_n$$

$\cos 90° = 0$

$\cos 0° = 1$

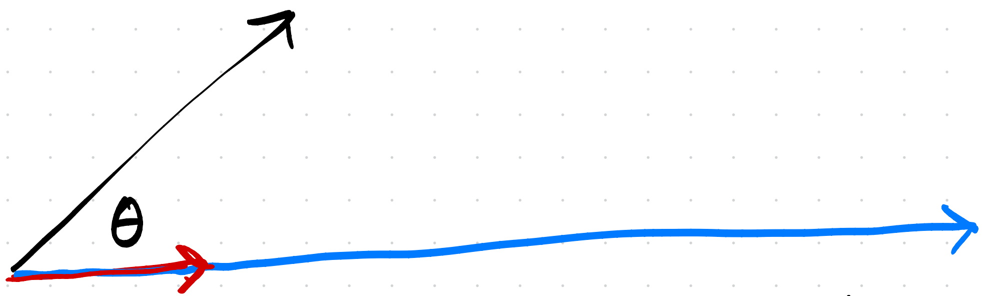
$\cos 180° = -1$

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta$$

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}$$

$$-1 \le \cos\theta \le 1$$

"cosine similarity" of two vectors



$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|} = \underbrace{\left(\frac{\vec{u}}{\|\vec{u}\|}\right)}\cdot\underbrace{\left(\frac{\vec{v}}{\|\vec{v}\|}\right)}$$

unit vectors,
because their
length is 1!

Two important inequalities involving norms

① Triangle inequality

$\vec{u}, \vec{v} \in \mathbb{R}^n$

$$\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$$

applies to all norms:

L_2, L_1, L_∞, ...

no side can be longer than sum of other 2

② Cauchy-Schwarz → only for L_2! (default norm)

$$|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\| \|\vec{v}\|$$

Activity : Using Cauchy-Schwarz, prove that for any positive a and b, geometric mean $\leq$ arithmetic mean

$$\sqrt{ab} \leq \frac{a+b}{2}$$

$\vec{u}, \vec{v} \in \mathbb{R}^n$

$|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\| \, \|\vec{v}\|$

$$\vec{u} = \begin{bmatrix} \sqrt{a} \\ \sqrt{b} \end{bmatrix} \quad \vec{v} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$

$\sqrt{ab} < \frac{a+b}{2}$

want to show

$$\underbrace{|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\| \, \|\vec{v}\|}$$

$$\vec{u} = \begin{bmatrix} \sqrt{a} \\ \sqrt{b} \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} \sqrt{b} \\ \sqrt{a} \end{bmatrix}$$

$$\left| \sqrt{a}\sqrt{b} + \sqrt{b}\sqrt{a} \right| \leq \sqrt{a+b}\,\sqrt{b+a}$$

$$2\sqrt{ab} \leq a+b$$

$$\sqrt{ab} \leq \frac{a+b}{2}$$

very common MT1 Q.