EECS 245, Spring 2026

LEC 4 Projections, Span, and Linear Independence

→ Read: Ch. 3.4,
Ch. 4.1, 4.2

Agenda

- Recap problem from last class
- The "approximation problem"
  - → Motivation
  - → Solution: Orthogonal Projections
  - → Examples

Ch. 3.4

- Span and linear independence

4.1–4.2

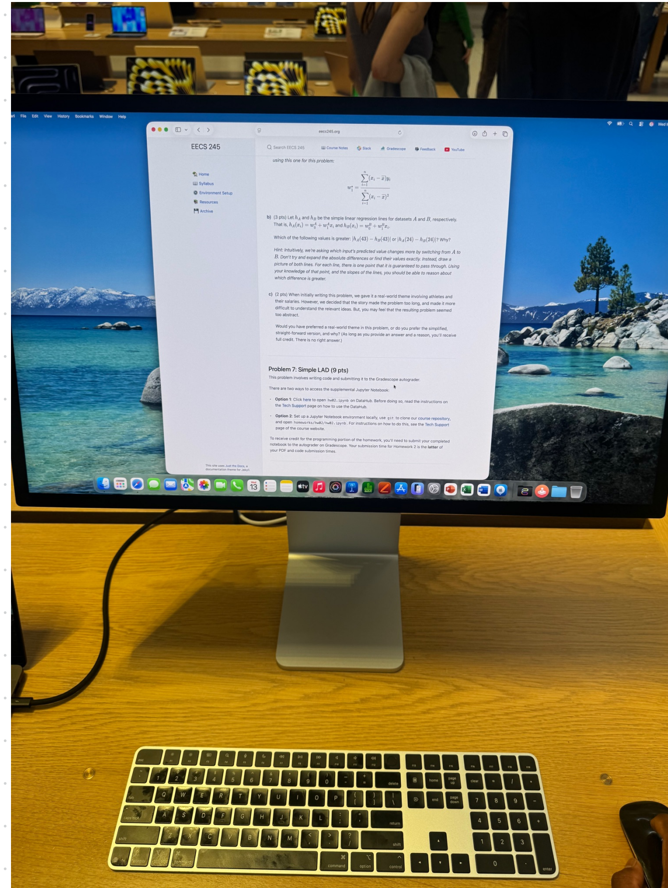
Announcements

- Lab 3 solutions are up
- HW 3 due Sunday
  - → HW 2 sol's will come out Sat morning
- HW 4 due Wednesday, out by tomorrow
- Midterm 1 is next Friday from 1-3PM in 1690 BBB

scope: Lec 1–5 (Ch. 1–4), Labs 1–5, HW 1–5

→ Look at past exams!


BOERUM PLACE
cobble hill


EECS 245
Problem 7: Simple LAD (9 pts)

Activity 4

Ch. 3.3

$\cos(-\theta) = \cos\theta$

$\cos(180-\theta) = -\cos\theta$

Suppose the dot product of $\vec{x}$ and $\vec{y}$ is 10, and the angle between $\vec{x}$ and $\vec{y}$ is 30°.

1. What is the dot product of $2\vec{x}$ and $3\vec{y}$? $6 \cdot 10 = 60$
2. What is the angle between $2\vec{x}$ and $3\vec{y}$? $\theta = 30^\circ$
3. What is the angle between $2\vec{x}$ and $-3\vec{y}$? $\theta = 180^\circ - 30^\circ$ $= 150^\circ$

$\vec{x} \cdot \vec{y} = 10$

$\theta = 30^\circ$

$(2\vec{x}) \cdot (3\vec{y}) = 6(\vec{x} \cdot \vec{y})$

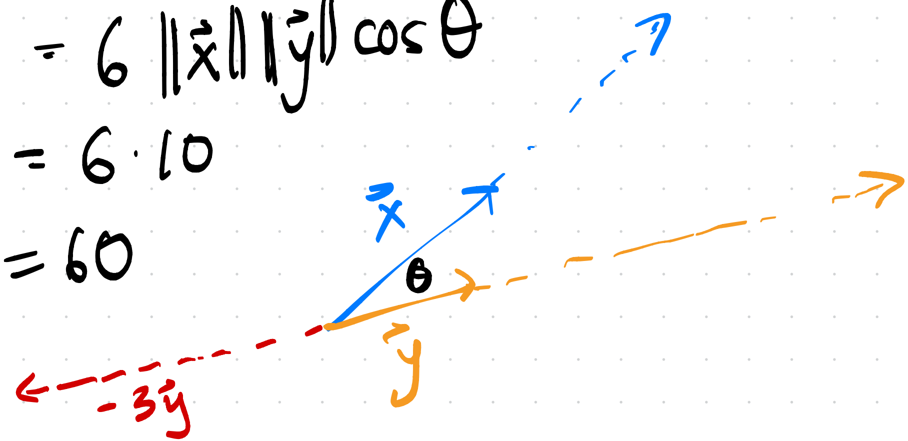
$$\begin{bmatrix} 2x_1 \\ 2x_2 \\ \vdots \\ 2x_n \end{bmatrix} \cdot \begin{bmatrix} 3y_1 \\ 3y_2 \\ \vdots \\ 3y_n \end{bmatrix}$$

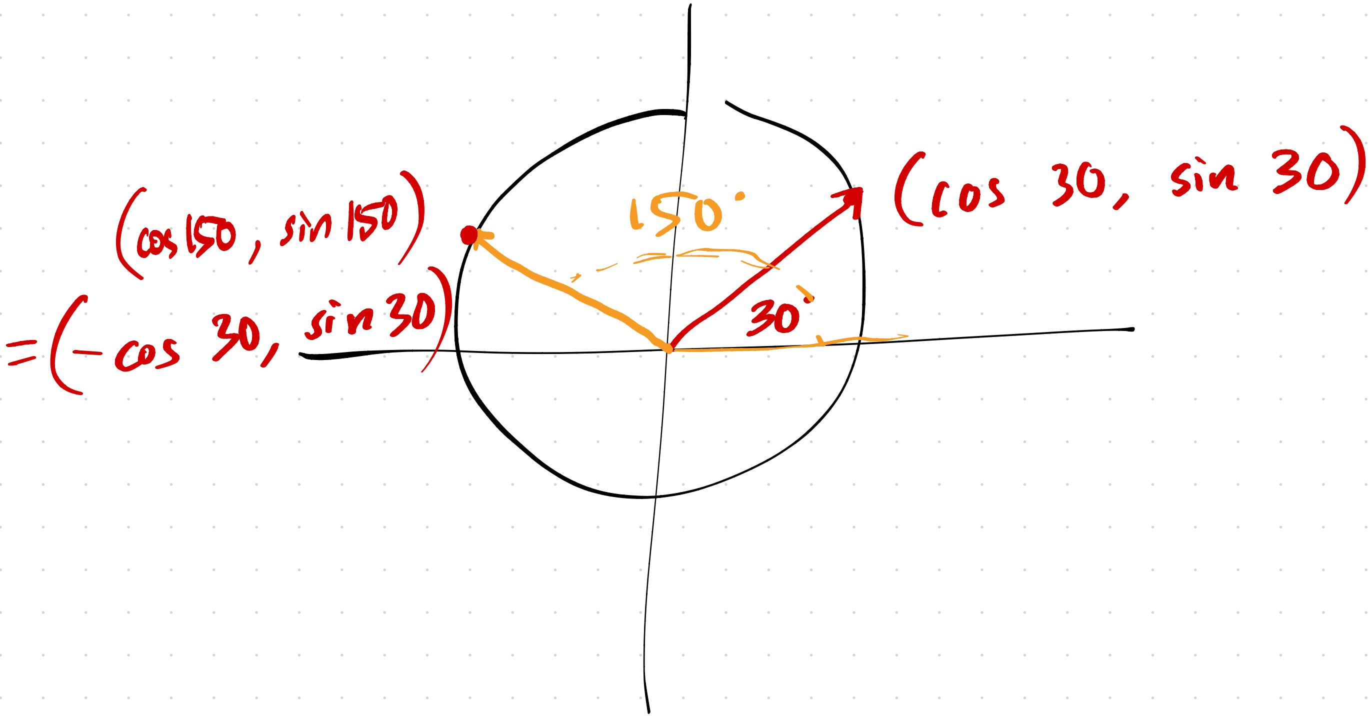
$= (2)(3)(x_1y_1 + x_2y_2 + x_3y_3 + \cdots + x_ny_n)$

$= 6\,\vec{x} \cdot \vec{y}$

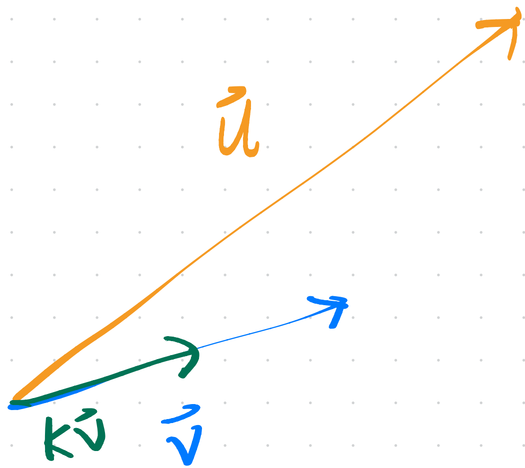
$$\vec{x} \cdot \vec{y} = \|\vec{x}\| \, \|\vec{y}\| \cos\theta$$

$$\begin{aligned}(2\vec{x}) \cdot (3\vec{y}) &= \|2\vec{x}\| \, \|3\vec{y}\| \cos\theta \\ &= (2\|\vec{x}\|)(3\|\vec{y}\|) \cos\theta \\ &= 6 \|\vec{x}\| \, \|\vec{y}\| \cos\theta \\ &= 6 \cdot 10 \\ &= 60\end{aligned}$$

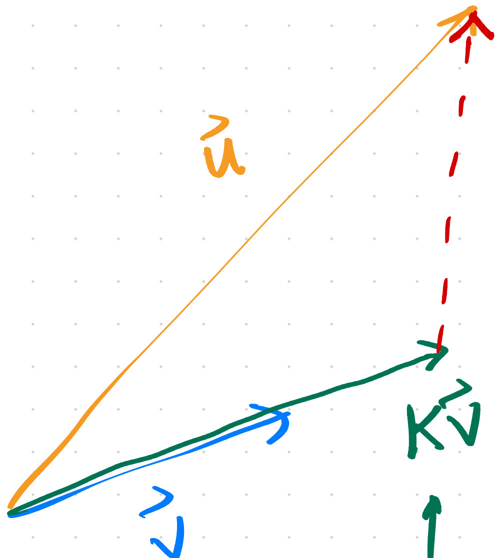



(cos 150, sin 150)
= (- cos 30, sin 30)
150°
30°
(cos 30, sin 30)

Chapter 3.4 "Approximation Problem"



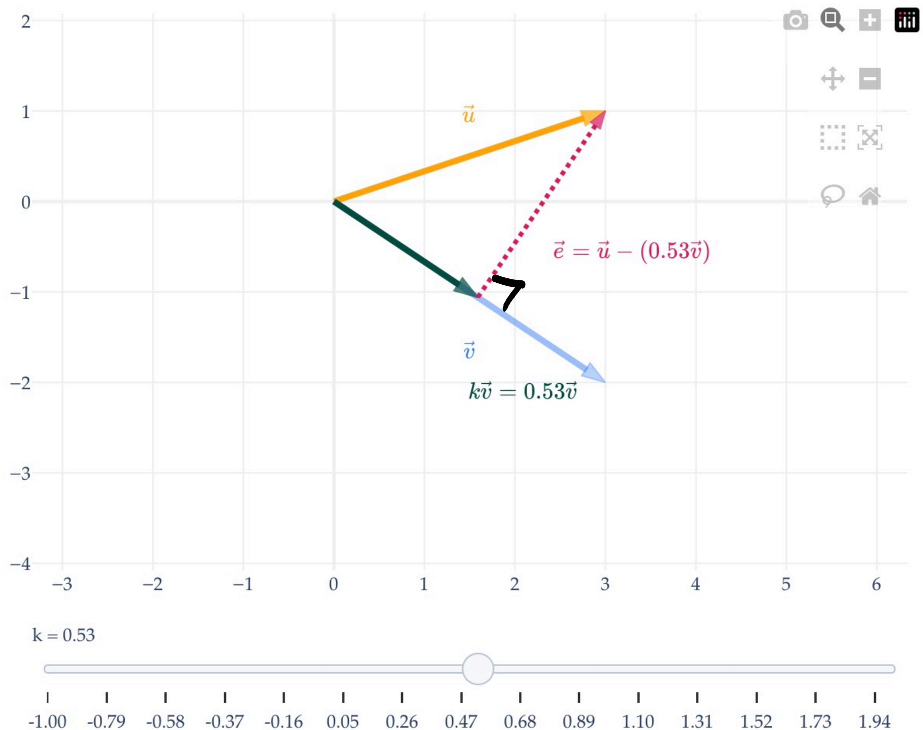
Q: Among all scalar multiples of $\vec{v}$, i.e. all vectors of the form $k\vec{v}$, where $k \in \mathbb{R}$, which is "closest" to $\vec{u}$?



$\vec{e} = \vec{u} - k\vec{v}$ is the "error vector"

→ Goal: Pick k such that $\|\vec{e}\|$ is as short as possible!

in this picture, $k \approx 1.6$



Guess:

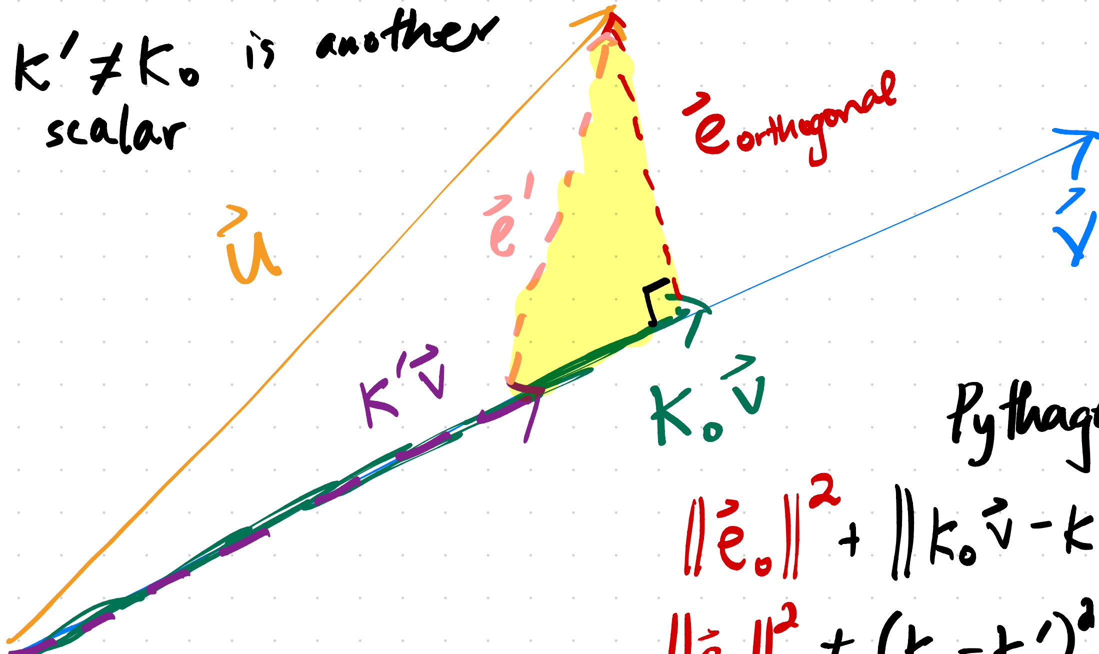
the shortest possible error vector is the one that is **orthogonal** to $\vec{v}$ (and $k\vec{v}$)!

HW 3:

$$f(k) = \underbrace{\|\vec{u} - k\vec{v}\|^2}_{\text{minimize}} = (\vec{u} - k\vec{v}) \cdot (\vec{u} - k\vec{v})$$

Suppose k_0 is chosen such that $\vec{e}_0 = \vec{u} - k_0 \vec{v}$ is orthogonal to $\vec{v}$

and $k' \neq k_0$ is another scalar

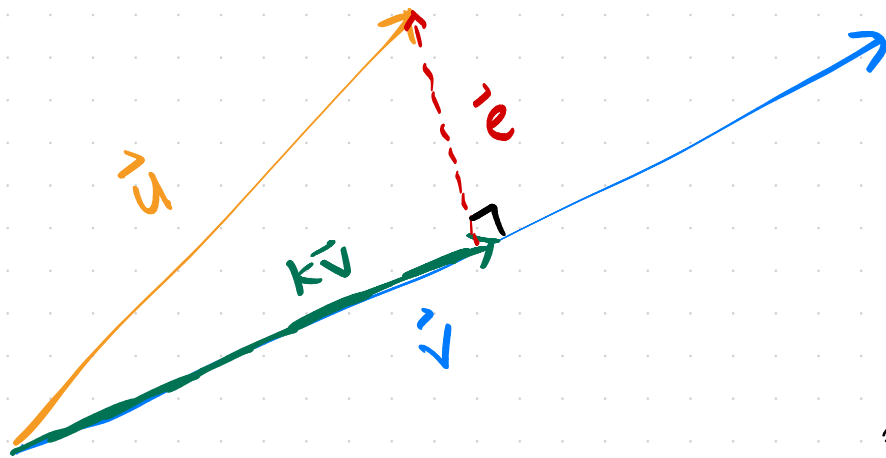


Pythagorean theorem:

$$\|\vec{e}_0\|^2 + \|k_0\vec{v} - k'\vec{v}\|^2 = \|\vec{e}\,'\|^2$$

$$\|\vec{e}_0\|^2 + \underbrace{(k_0 - k')^2 \|\vec{v}\|^2}_{>0\,!} = \|\vec{e}\,'\|^2$$

So, $\|\vec{e}_0\|^2 < \|\vec{e}\,'\|^2$



$\vec{e} = \vec{u} - k\vec{v}$

Punchline: In order to make $\vec{e}$, the error vector, as short as possible, pick k such that $\vec{e}$ is orthogonal to $\vec{v}$!

Task: Find k^*, i.e. the k that makes $\vec{e}$ orthogonal to $\vec{v}$.

Need $\vec{v}$ to be orthogonal to $\vec{u} - k\vec{v}$

<u>How?</u> Their dot product must be 0.

$$\vec{v} \cdot (\vec{u} - k\vec{v}) = 0$$

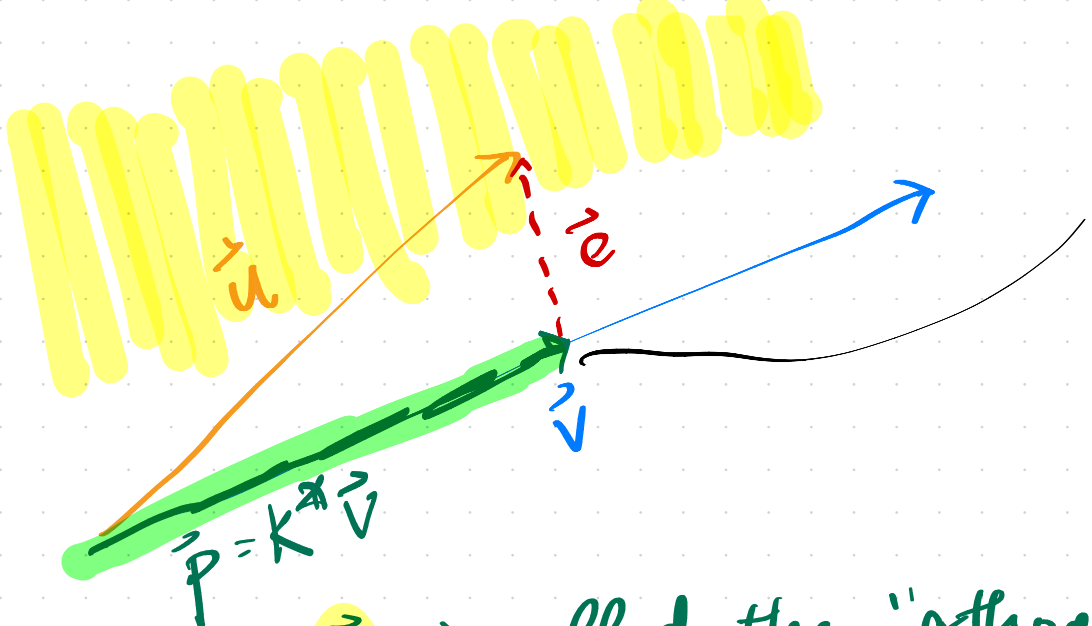
$$\vec{v} \cdot \vec{u} - \vec{v} \cdot (k\vec{v}) = 0$$

$$\vec{u} \cdot \vec{v} - k(\vec{v} \cdot \vec{v}) = 0$$

$$\vec{u} \cdot \vec{v} = k(\vec{v} \cdot \vec{v})$$

optimal value of k!

$$\boxed{k^* = \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} = \frac{\|\vec{u}\| \cos\theta}{\|\vec{v}\|}$$



"shadow" of $\vec{u}$ onto $\vec{v}$

$\vec{p}$ is called the "orthogonal projection" of $\vec{u}$ onto $\vec{v}$

Example: $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

① Find projection of $\vec{u}$ onto $\vec{v}$

② Find error vector, verify it is orthogonal (to who?)

③ Find $\|\vec{e}\|$ (length of error vec)

① $k^* = \dfrac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}$

$= \dfrac{5}{3}$

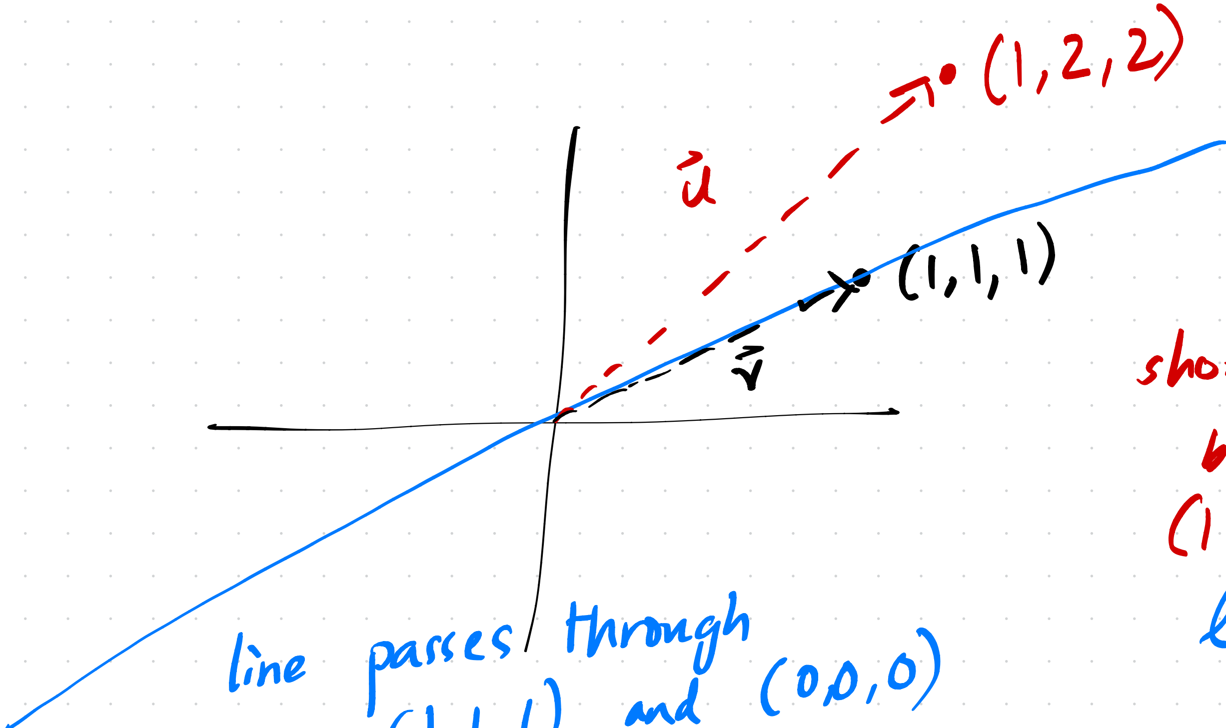
$\vec{p} = \dfrac{5}{3} \vec{v} = \begin{bmatrix} 5/3 \\ 5/3 \\ 5/3 \end{bmatrix}$

② $\vec{e} = \vec{u} - \vec{p} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 5/3 \\ 5/3 \\ 5/3 \end{bmatrix} = \begin{bmatrix} -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}$

$\vec{e} \cdot \vec{v} = \begin{bmatrix} -2/3 \\ 1/3 \\ 1/3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0$ ✓

③ $\|\vec{e}\| = \sqrt{\left(2/3\right)^2 + \left(1/3\right)^2 + \left(1/3\right)^2}$

$= \sqrt{6/9} = \dfrac{\sqrt{6}}{3}$



line passes through $(1,1,1)$ and $(0,0,0)$

shortest distance between $(1,2,2)$ and line?

$$\frac{\sqrt{6}}{3} \; !$$

same answer

$\vec{u} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

$\vec{v}_1 = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$

$\vec{v}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

Task: Write $\vec{u}$ as a linear combination of $\vec{v}_1$ and $\vec{v}_2$

$$\vec{u} = \frac{1}{5}\vec{v}_1 + \frac{9}{5}\vec{v}_2$$

$$\begin{bmatrix} 3 \\ 5 \end{bmatrix} = a\begin{bmatrix} 6 \\ -2 \end{bmatrix} + b\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$3 = 6a + b \rightarrow$$

$$15 = -6a + 9b$$

$$18 = 10b \Rightarrow b = \frac{18}{10} = \frac{9}{5}$$

$$6a = 3 - b = 3 - \frac{9}{5} = \frac{6}{5}$$

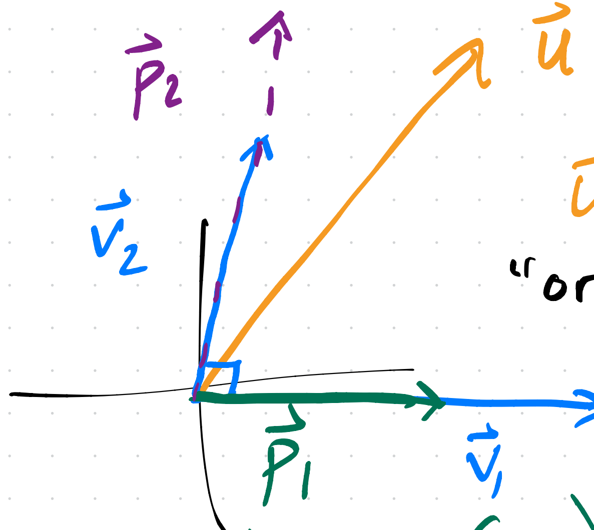
$$\Rightarrow a = \frac{1}{5}$$

$$\vec{u} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$\vec{v}_1 = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$$

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Note: $\vec{v}_1$ and $\vec{v}_2$ are orthogonal!



$$\vec{u} = \vec{p}_1 + \vec{p}_2$$

"orthogonal decomposition" of $\vec{u}$

$$\vec{u} = \frac{1}{5}\vec{v}_1 + \frac{9}{5}\vec{v}_2$$

$$\vec{p}_1 = \left(\frac{\vec{u} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\right)\vec{v}_1 = \left(\frac{8}{40}\right)\vec{v}_1 = \frac{1}{5}\vec{v}_1$$

$$\vec{p}_2 = \left(\frac{\vec{u} \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\right)\vec{v}_2 = \left(\frac{18}{10}\right)\vec{v}_2 = \frac{9}{5}\vec{v}_2$$

Suppose $\vec{v}_1, \vec{v}_2$ orthogonal,
and that $\vec{u}$ can be written
as a linear combination of both

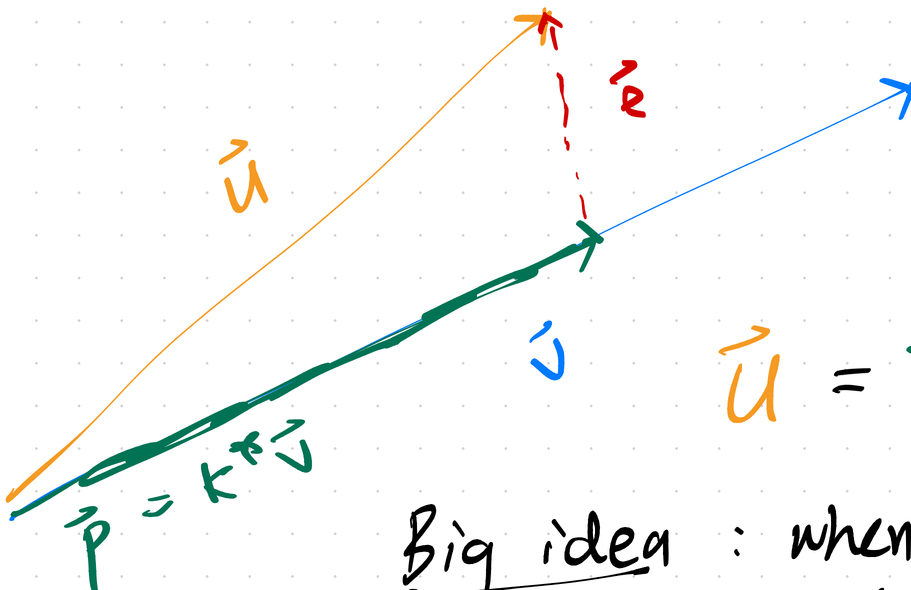
$$\vec{u} = \underline{a}\,\vec{v}_1 + \underline{b}\,\vec{v}_2$$

idea: dot product of both sides with $\vec{v}_1$!

$$\begin{aligned}\vec{u}\cdot\vec{v}_1 &= (a\vec{v}_1 + b\vec{v}_2)\cdot\vec{v}_1 \\ &= a(\vec{v}_1\cdot\vec{v}_1) + \cancel{b(\vec{v}_2\cdot\vec{v}_1)} \\ &= a(\vec{v}_1\cdot\vec{v}_1)\end{aligned}$$

$$\Rightarrow \boxed{a = \frac{\vec{u}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}}$$

2:42



$$\vec{u} = \vec{p} + \vec{e}$$

Big idea: when writing $\vec{u}$ as a linear combination of some $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d$, ideal if the $\vec{v}_i$'s are orthogonal!!!

Chapter 4.1

"building blocks"

The **span** of the vectors $\underbrace{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_d}_{\text{"spanning set"}} \in \mathbb{R}^n$

is the set of all possible linear combinations of them

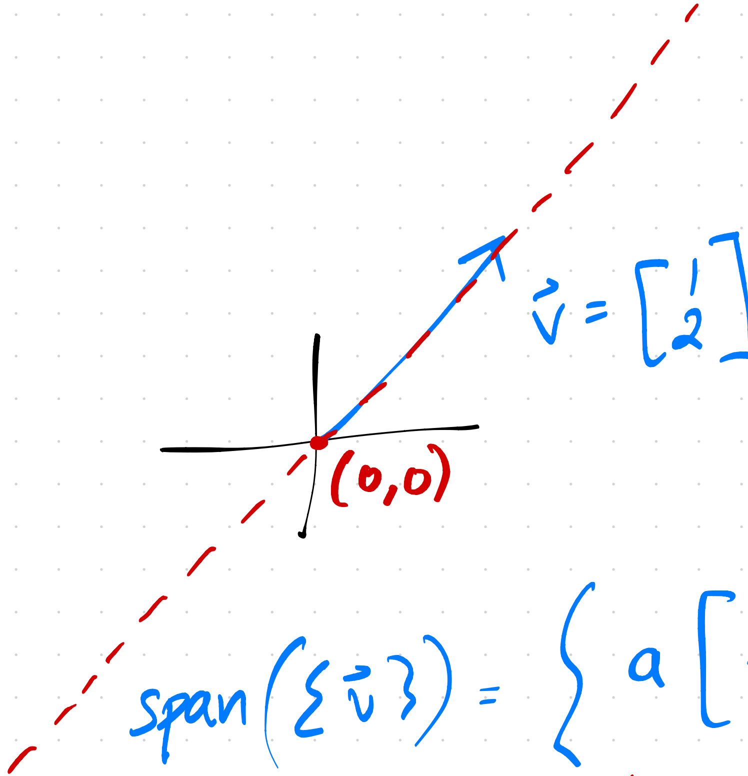
$$\text{span}(\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_d\}) = \underbrace{\{a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_d\vec{v}_d \mid a_1, a_2, \dots, a_d \in \mathbb{R}\}}_{\text{set of all possible lin. combs.}}$$

1 vector

$\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\mathbb{R}^2$

2 dimensional example



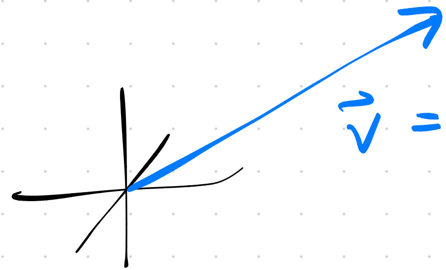
$$\operatorname{span}(\{\vec{v}\}) = \left\{ a \begin{bmatrix} 1 \\ 2 \end{bmatrix} \;\middle|\; a \in \mathbb{R} \right\}$$

= a line through (0,0) and (1,2)

another formula: $y = 2x$

$$\left\{ \begin{bmatrix} x \\ y \end{bmatrix} \;\middle|\; y = 2x \right\}$$

1 vector in $\mathbb{R}^3$?



$$\vec{v} = \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix}$$

$$\operatorname{span}(\{\vec{v}\}) = \underbrace{\left\{ a \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix} \,\middle|\, a \in \mathbb{R} \right\}}_{\text{parametric form of a line in } \mathbb{R}^3}$$

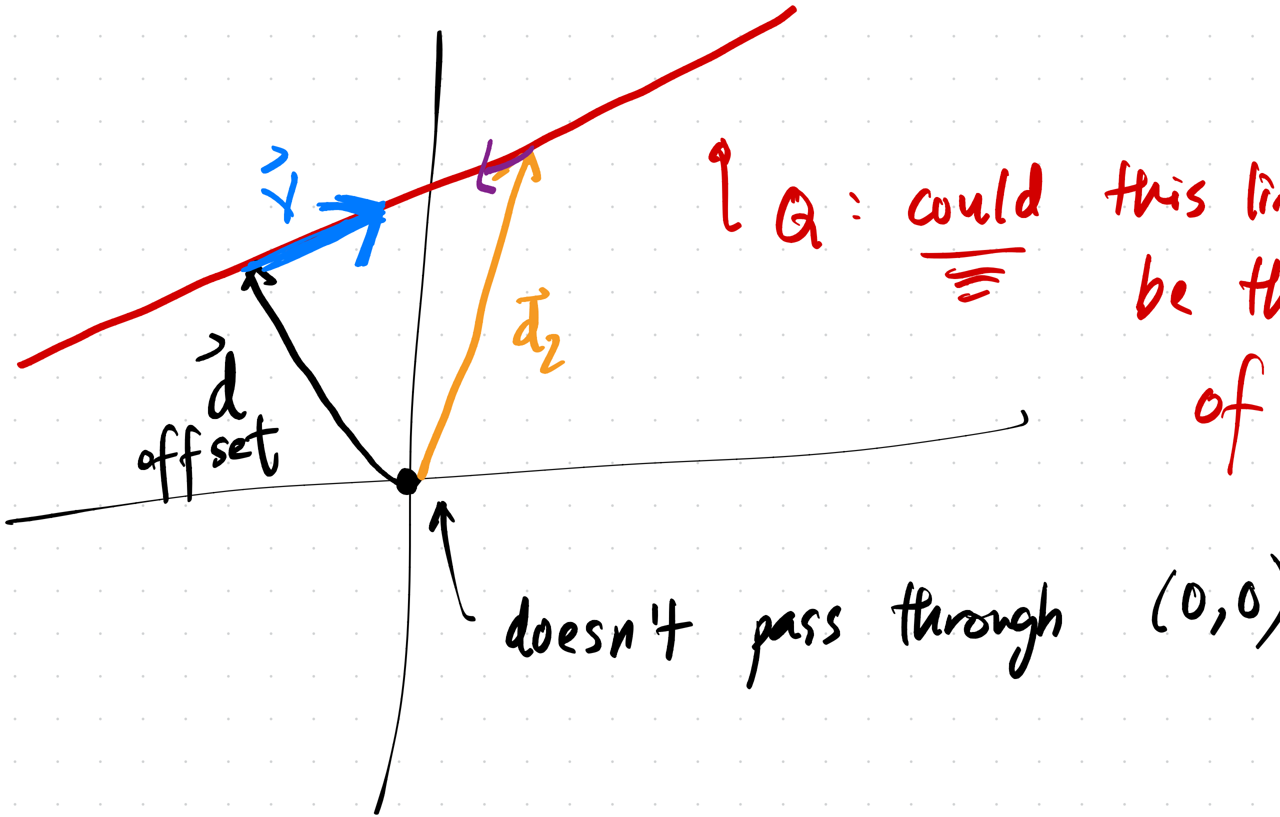
1 vector in $\mathbb{R}^7$

$$\vec{v} = \begin{bmatrix} \pi \\ 2.73 \\ 1998 \\ 12 \\ 0 \\ 0 \\ -132 \end{bmatrix}$$

$\text{span}(\{\vec{v}\})$ is a line in $\mathbb{R}^7$

1-dimensional "slice" of $\mathbb{R}^7$

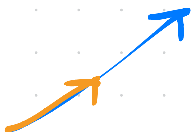
$L: \vec{d} + t\vec{v}, \quad t \in \mathbb{R}$ parametric form



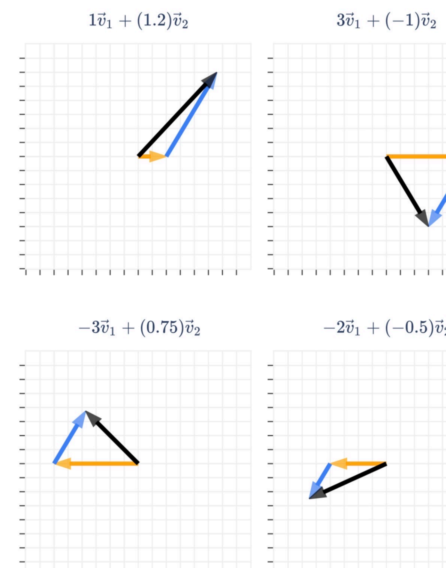
Q: could this line be the span of a vector?

2 vectors   start with $\mathbb{R}^2$

① both $\vec{0}$

②  they are collinear : span is a line!

③ the two vectors are not collinear in $\mathbb{R}^2$, their span is all of $\mathbb{R}^2$!



$\mathbb{R}^5$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ -\pi^3 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 7 \\ 12 \\ 13 \\ -101 \\ 4 \end{bmatrix}$$

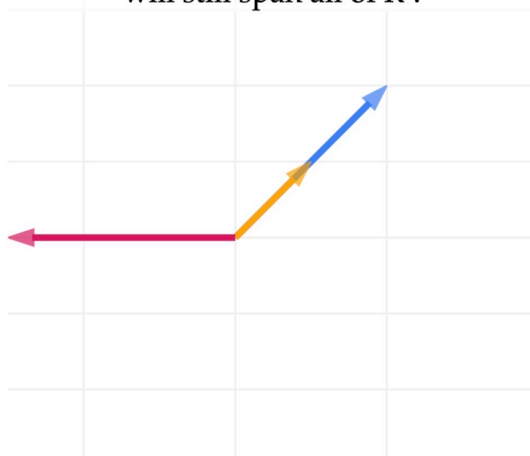
$$\operatorname{span}\left(\{\vec{v}_1, \vec{v}_2\}\right) = \underbrace{\left\{ a_1 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ -\pi^3 \end{bmatrix} + a_2 \begin{bmatrix} 7 \\ 12 \\ 13 \\ -101 \\ 4 \end{bmatrix} \;\middle|\; a_1, a_2 \in \mathbb{R} \right\}}_{\text{2-dimensional subspace of } \mathbb{R}^5}$$

subspace: flat "slice"

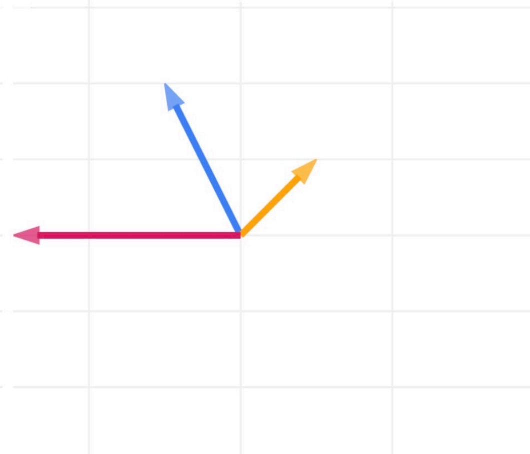
3 vectors

$\mathbb{R}^2$

The blue and orange vectors are redundant.
Remove either one of them,
and the remaining 2 vectors
will still span all of R².



Any 2 of these 3 vectors
span all of R².
One is redundant.



3 vectors, in $\mathbb{R}^3$ could span:

1. $\{\vec{0}\}$ — 0-dimensional subspace of $\mathbb{R}^3$
2. line — 1-dimensional subspace of $\mathbb{R}^3$

3. plane — 2-dimensional subspace of $\mathbb{R}^3$
4. all of $\mathbb{R}^3$! — 3-dimensional subspace of $\mathbb{R}^3$

In general,

$\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d \in \mathbb{R}^n$

d vectors, all in $\mathbb{R}^n$

# of vectors

span a between 0 and $\underline{\min(n, d)}$ - dimensional

subspace of $\mathbb{R}^n$.

dimension of the universe the vectors live in

2 vectors in $\mathbb{R}^4$ max dim of span: 2

4 vectors in $\mathbb{R}^2$ max dim of span: 2

Activity: Find 6 vectors in $\mathbb{R}^4$ that span a 3-dimensional subspace of $\mathbb{R}^4$

one possible solution

$$\Bigg\{ \underbrace{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}}_{\text{independent}}, \underbrace{\begin{bmatrix}2\\0\\0\\0\end{bmatrix}, \begin{bmatrix}3\\6\\0\\0\end{bmatrix}, \begin{bmatrix}4\\0\\0\\0\end{bmatrix}}_{\text{can make with first 3}} \Bigg\}$$

another possible

$$\Bigg\{ \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}2\\0\\0\\2\end{bmatrix}, \begin{bmatrix}15\\-17\\-17\\1342\end{bmatrix} \Bigg\}$$

Linear independence (Chapter 4.2)

Vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d$ are linearly independent

if (1) none of the vectors can be written as a linear combination of the others

same!

(2) the only solution to

$$a_1 \vec{v}_1 + a_2 \vec{v}_2 + \cdots + a_d \vec{v}_d = \vec{0}$$

zero vector

is $a_1 = a_2 = \cdots = a_d = 0$ "trivial"

otherwise, the vectors are linearly dependent

How are these 2 definitions related?

suppose 3 vectors, $\vec{v}_1, \vec{v}_2, \vec{v}_3$

satisfy $\vec{v}_3 = 2\vec{v}_1 - 4\vec{v}_2$

these $\vec{v}_i$'s are not linearly independent,

so they are linearly dependent

rearrange: $2\vec{v}_1 - 4\vec{v}_2 - \vec{v}_3 = \vec{0}$

$\Rightarrow$ can make $\vec{0}$ using a "non-trivial" linear combination of $\vec{v}_i$'s (at least one $a_i \neq 0$)

```
given v_1, v_2, ..., v_d
initialize linearly independent set S = {v_1}
for i = 2 to d:
    if v_i is not a linear combination of S:
        add v_i to S
```

Chapter 4.2

Activity 2

Ch 4.4 talks about to find eq'n of plane in $\mathbb{R}^3$ in $ax+by+cz=0$ "cross product"

To recap what we've covered in this section, answer the following questions.

1. Can any three vectors in $\mathbb{R}^2$ be linearly independent? no! max # is 2
2. **Must** any two vectors in $\mathbb{R}^2$ be linearly independent? no!
3. If two vectors in $\mathbb{R}^3$ are linearly independent, what do they span? plane
4. If three vectors in $\mathbb{R}^3$ are linearly independent, what do they span? all of $\mathbb{R}^3$
5. Given d vectors in $\mathbb{R}^n$, what must be true about d and n for it to be possible for the vectors to be linearly independent? $d \leq n$

Solutions