EECS 245, Spring 2026

LEC 5 Vector Spaces and Subspaces

→ Read: Ch. 4.2-4.3

Agenda

- Recap: Linear independence
- Recap: Equation of a plane
- Formal definitions:
  - vector space
  - subspace
  - basis
  - dimension

Ch. 4.3

Announcements

- MT 1 on Friday: see eecs245.org/mt1 for details
- HW 4 + Lab 5 due tomorrow; no slip days on HW 4!
- HW 1-3 + Lab 1-4 solutions posted
- Remember: everyone needs 14 engagement points! Come to OH!

Chapter 4.2

Activity 3

1. Suppose we have 8 vectors in $\mathbb{R}^{17}$.
   - Could they be linearly independent? yes, they could, but not guaranteed
   - Could they span all of $\mathbb{R}^{17}$? no: need at least 17 vectors to span all of $\mathbb{R}^{17}$
2. Suppose we have 17 vectors in $\mathbb{R}^{8}$.
   - Could they be linearly independent? no
   - Could they span all of $\mathbb{R}^{8}$? they could, but not guaranteed

Linearly independent?

$\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d \in \mathbb{R}^n$ are linearly independent

if

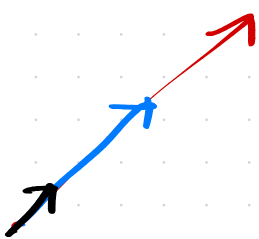
① the only linear combination of $\vec{v}_i$'s that makes $\vec{0}$ is

$$0\vec{v}_1 + 0\vec{v}_2 + \cdots + 0\vec{v}_d = \vec{0}$$

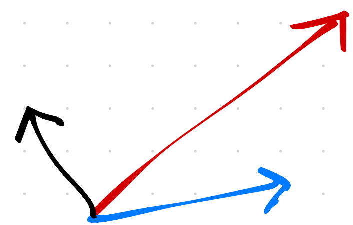
otherwise, no linear combination $= \vec{0}$

② none of the $\vec{v}_i$'s can be written as a linear combination of the other $\vec{v}_i$'s collinear just means they point in same direction

e.g. 3 vectors in $\mathbb{R}^2$



collinear



not collinear, but still linearly dependent

Activity 7

Chapter 4.4 (linked in various labs and homeworks)

1. Find the equation, in standard form, of the plane spanned by $\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.

   Why did you not need to compute the cross product?

2. Find the equation, in standard form, of the plane spanned by $\begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$.

next page

$z = 0$ since both vectors live in xy plane

$$\underbrace{\vec{u} = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}}_{\text{want plane spanned by these 2}}$$

standard form:

$$ax + by + cz = 0$$

goal: find a, b, c (d must be 0)

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

cross product

$$\vec{u} \times \vec{v} = \begin{bmatrix} 2\cdot 2 - 4\cdot 1 \\ 4\cdot(-1) - 3\cdot 2 \\ 3\cdot 1 - 2(-1) \end{bmatrix} = \begin{bmatrix} 0 = a \\ -10 = b \\ 5 = c \end{bmatrix}$$

eq'n:

$$0x - 10y + 5z = 0$$

$$\boxed{2y - z = 0}$$

$$z = 2y$$

final answer

3	-1
2	1
4	2
3	-1
2	1

Chapter 4.3

see 4.3 for "properties" of a VS

$\mathbb{R}^n$ "Euclidean vector"

Vector space is a set of objects, V, (called vectors) that support 2 operations:

1. Addition $\vec{u}, \vec{v} \in V \Rightarrow \vec{u} + \vec{v} \in V$

2. Scalar multiplication $\vec{v} \in V,\ c \in \mathbb{R} \Rightarrow c\vec{v} \in V$

$\Rightarrow$ element of a vector space support linear combinations!

$c, d \in \mathbb{R}$

$\vec{u}, \vec{v} \in V \Rightarrow \underbrace{c\vec{u} + d\vec{v} \in V}$

this is what it means for a vector space to support linear combinations e.g. $V =$ set of polynomials with degree ≤ 3

$$u(x) = 2x^3 - 15x^2 + 3x + 94$$

$$v(x) = x^2 + 3$$

need $\leq$!

$$\begin{array}{r} 2x^3 + 5 \\ + \quad -2x^3 - x^2 \\ \hline 5 - x^2 \end{array}$$

not deg 3!

Subspace

this means V is a set, too

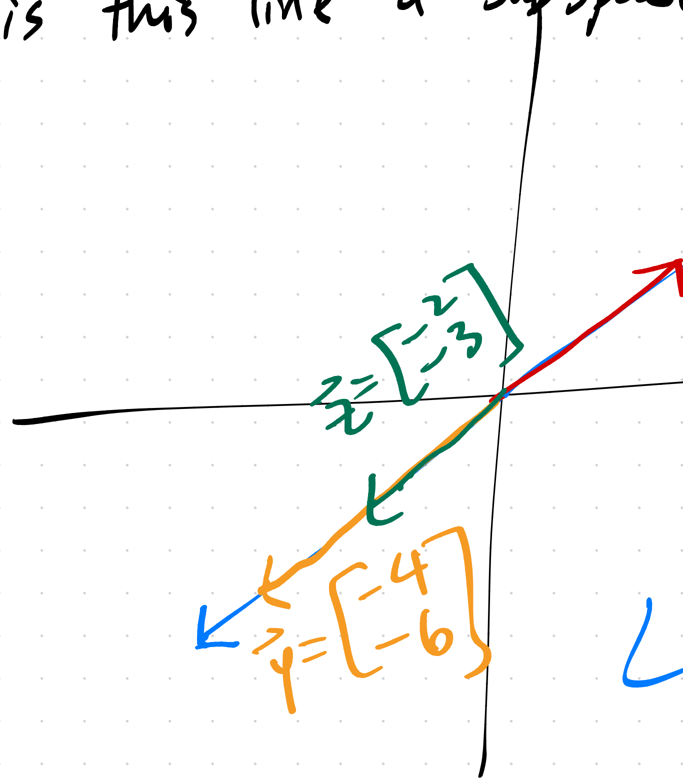
Suppose V is a vector space, and that S is a subset of V. S is a subspace of V if:

1. $\vec{u}, \vec{v} \in S \Rightarrow \vec{u} + \vec{v} \in S$
2. $c \in \mathbb{R}, \vec{v} \in S \Rightarrow c\vec{v} \in S$
3. $\vec{0} \in S$

S itself is a vector space!

S is "closed" under addition + scalar multiplication e.g. a line through $(0,0)$ in $\mathbb{R}^2$

is this line a subspace? yes! if you pick any 2 vectors on the line and take any linear combination of them, it will also be on line



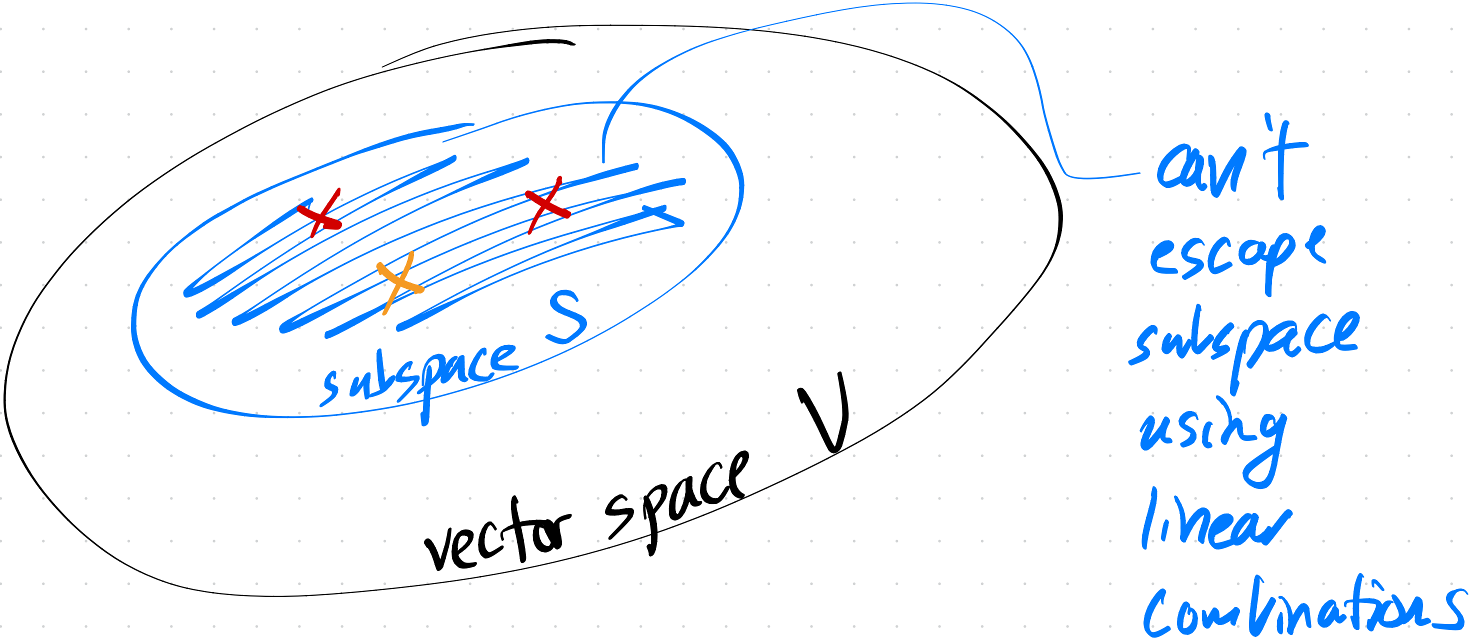
$\vec{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

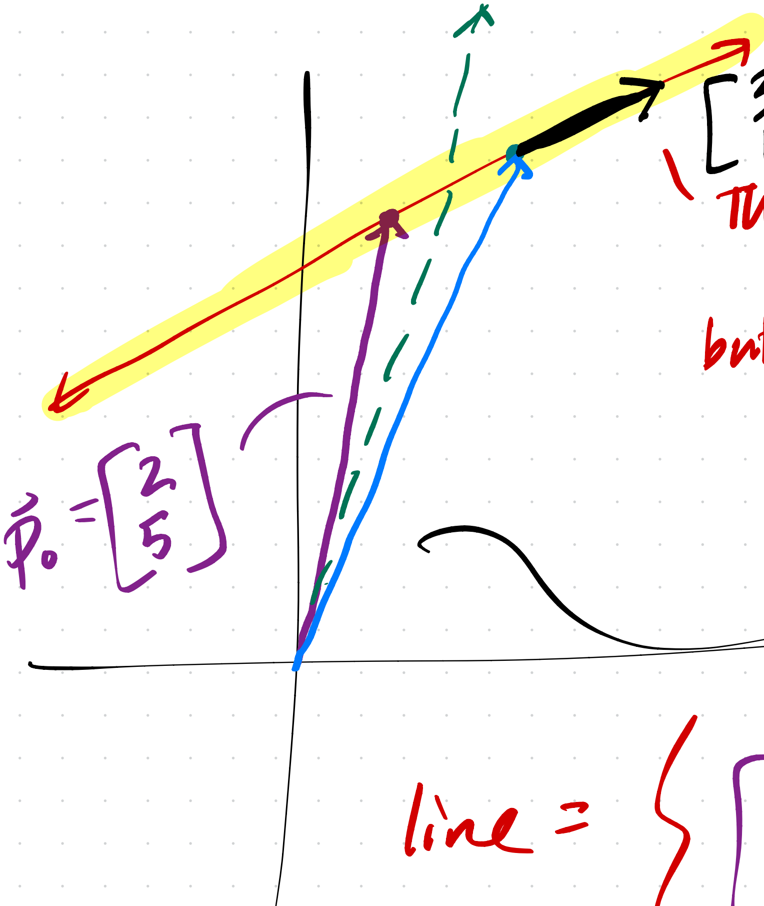
→ what if I add?

$\vec{z} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$

$$L = \text{span}\left(\left\{\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right\}\right) = \left\{ a\begin{bmatrix} 2 \\ 3 \end{bmatrix} \;\middle|\; a \in \mathbb{R} \right\}$$

subspace! ✓


subspace S
vector space V
can't escape subspace using linear combinations



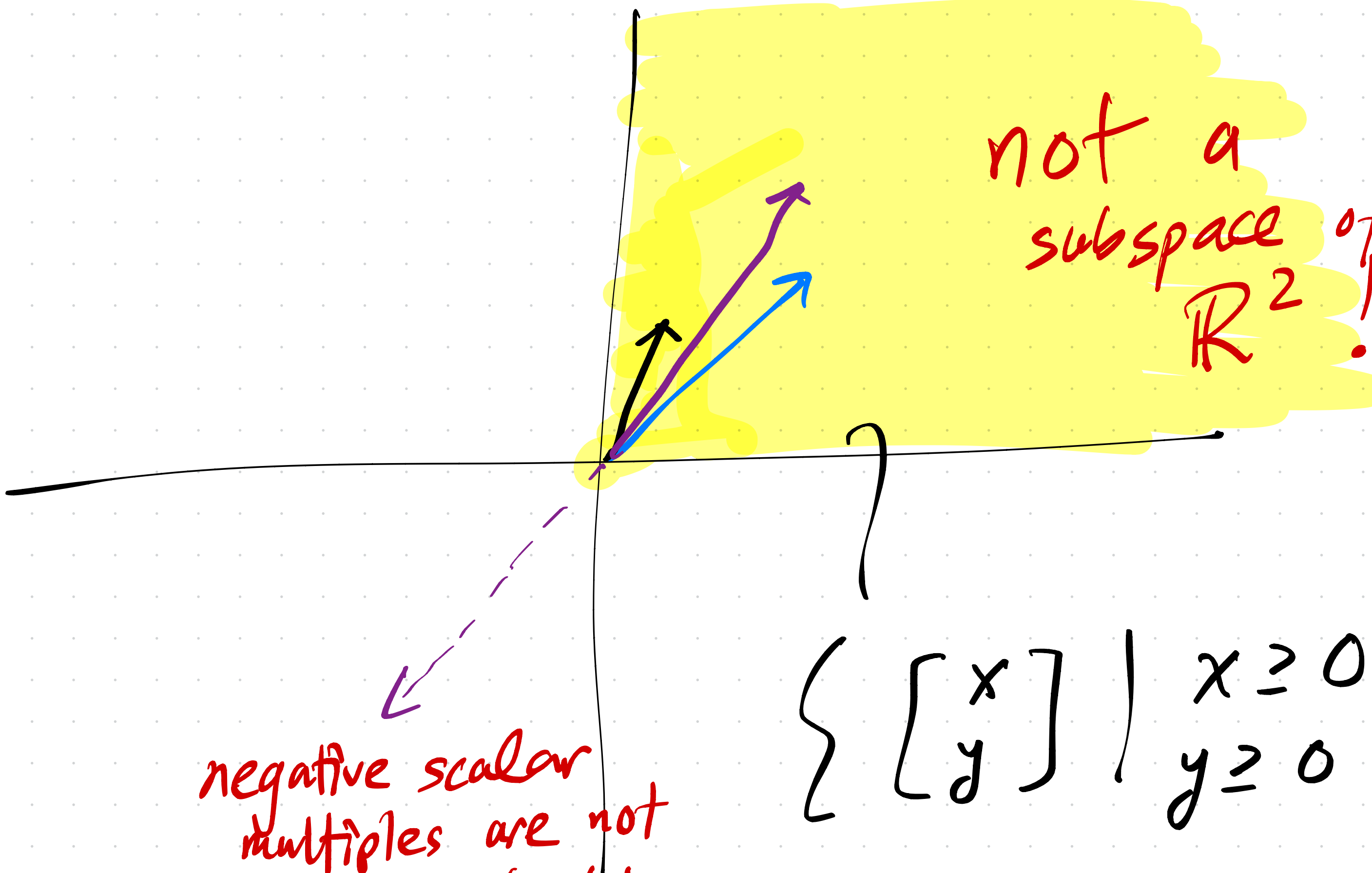
$\begin{bmatrix}3\\1\end{bmatrix}$

This line is a <u>subset</u> of $\mathbb{R}^2$

but not a subspace: doesn't contain $\begin{bmatrix}0\\0\end{bmatrix}$

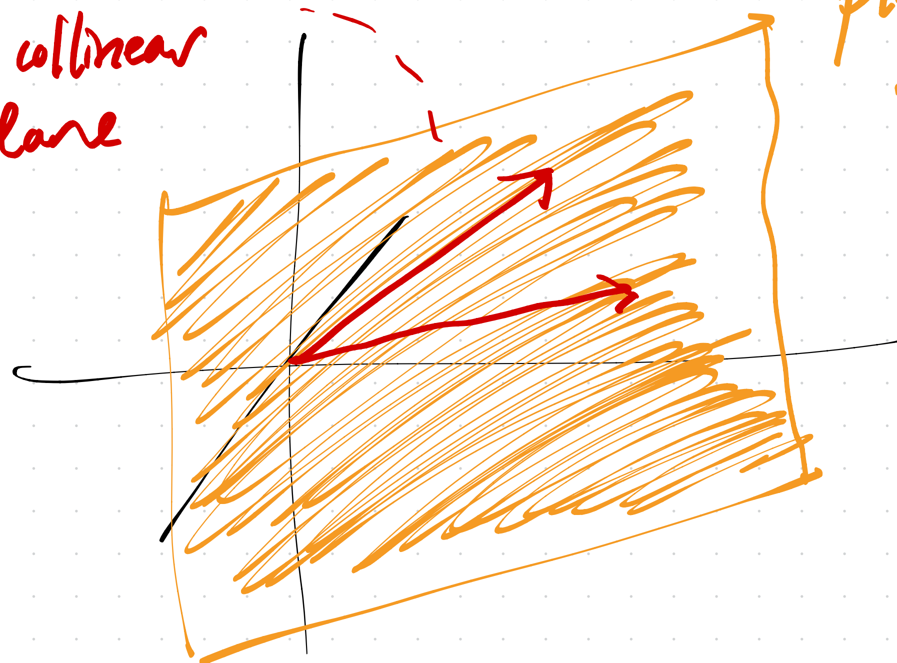
add these 2 vectors, result is not on line!

$$\text{line} = \underbrace{\left\{ \begin{bmatrix}2\\5\end{bmatrix} + a\begin{bmatrix}3\\1\end{bmatrix} \;\middle|\; a \in \mathbb{R} \right\}}_{\text{valid subset but } \underline{\text{not}} \text{ a subspace}}$$



$$\left\{ \begin{bmatrix} x \\ y \end{bmatrix} \;\middle|\; \begin{array}{l} x \geq 0, \\ y \geq 0 \end{array} \right\}$$

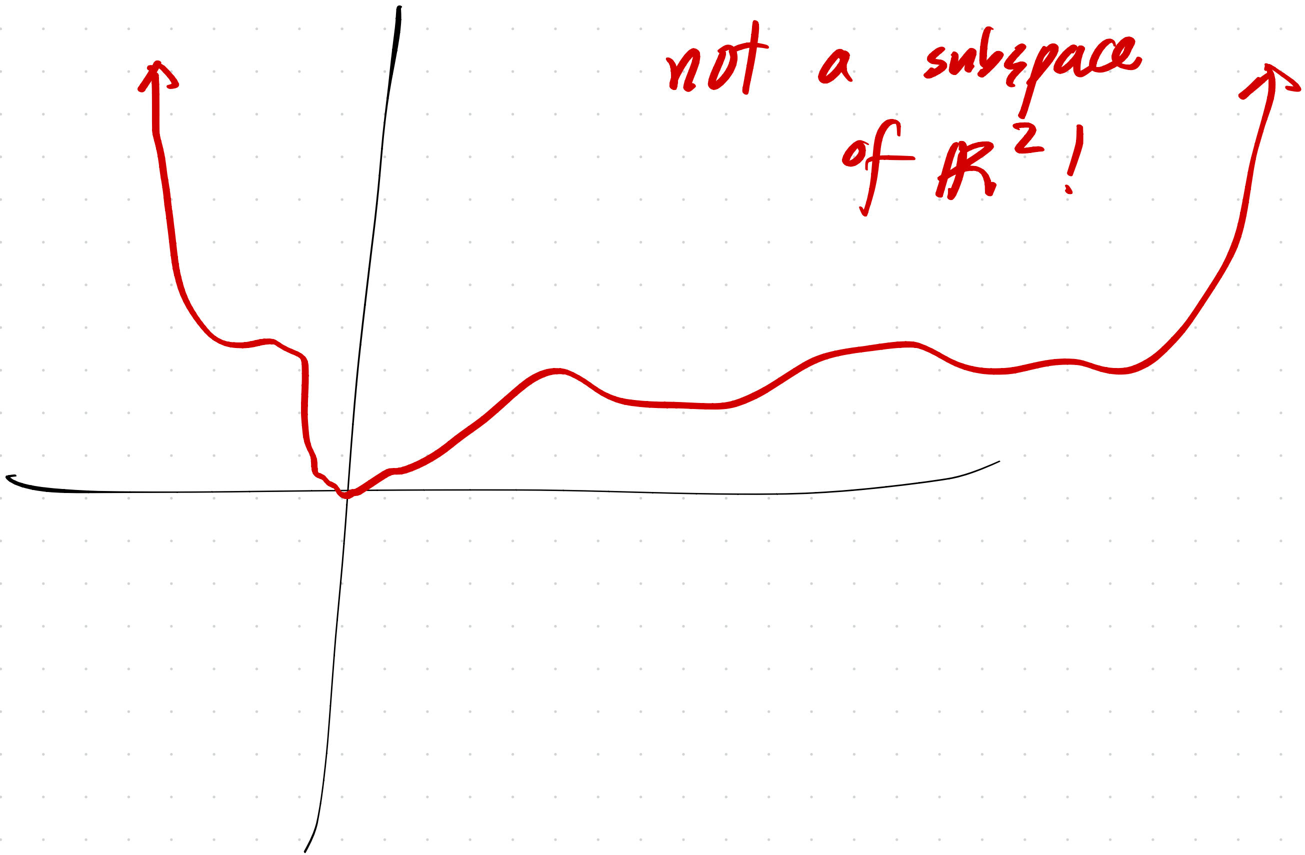
recap: any 2 vectors in $\mathbb{R}^3$ that aren't collinear span a plane



span = set of all possible linear combinations

subspace? yes!!!

can't escape it using linear combinations not a subspace of $\mathbb{R}^2$!



Big idea: no matter what vectors you have,

Span$\left(\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d\}\right)$ is a subspace

AND

all subspaces are spanned by some collection of vectors e.g. The set of vectors in $\mathbb{R}^5$ whose second and fourth components are equal to each other

① is this a subspace? how do you know?

② if it is, find vectors that span it

① subspace?

$\vec{0} \in S$ ✓

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

is the set closed under linear combinations?

$$\alpha \begin{bmatrix} a \\ b \\ c \\ b \\ d \end{bmatrix} + \beta \begin{bmatrix} e \\ f \\ g \\ f \\ h \end{bmatrix} = \begin{bmatrix} \alpha a + \beta e \\ \alpha b + \beta f \\ \alpha c + \beta g \\ \alpha b + \beta f \\ \alpha d + \beta h \end{bmatrix}$$

✓ ✓

so yes, S is a subspace!

This set is a subspace. Let's find vectors that span the subspace.

$$\begin{bmatrix} a \\ b \\ c \\ b \\ d \end{bmatrix} = a \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\vec{v}_1} + b \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\vec{v}_2} + c \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\vec{v}_3} + d \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{\vec{v}_4}$$

$\text{span}(\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\})$ is this subspace!

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

one possible answer

notice: both sets linearly ind. AND both have 4 vectors!

$$\left\{ \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 11 \\ 0 \\ 11 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 19 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \pi^3 \end{bmatrix} \right\}$$

also a possible answer

Basis

bases = plural of basis

A **basis** for a subspace S is a set of vectors that:

① are linearly independent, and

② span the whole subspace S

→ key idea: think of a basis as a "minimum set of building blocks" for S

→ a subspace has infinitely many bases?

Dimension

If S is a subspace then $\dim(S)$ is the number of vectors in every basis of S

Example: $2x - 3y + 4z = 0$

Key idea: 2 non-collinear vectors in $\mathbb{R}^3$ uniquely define plane

① Find 2 possible bases

② Find dimension $= 2$

$$\underbrace{\left\{ \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \right\}}_{\text{one possible basis}} \qquad \underbrace{\left\{ \begin{bmatrix} 0 \\ 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} \right\}}_{\text{another possible basis}}$$

Aside: "Standard" basis for $\mathbb{R}^n$

e.g. $\mathbb{R}^2$

$$\underbrace{\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} \pi \\ 152 \end{bmatrix} \right\}}_{\text{basis}} \qquad \underbrace{\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}}_{\text{orthogonal basis}}$$

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \quad \text{"standard" basis}$$

when we write vectors in $\mathbb{R}^2$, we're implicitly writing them as linear combinations of the standard basis

$$\begin{bmatrix} 7 \\ -3 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + (-3) \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Why do we care about linear independence?

→ when vectors are linearly independent, every linear combination of them can only be written in one way.

i.e. is unique!

→ who cares? we'd like to be able to interpret the coefficients in a linear combination when we get back to linear regression e.g. $\left\{ \underset{\vec{u}}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}, \underset{\vec{v}}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}, \underset{\vec{w}}{\begin{bmatrix} 3 \\ 4 \end{bmatrix}} \right\}$ $\rightarrow$ not a basis for $\mathbb{R}^2$, <u>but</u> they span $\mathbb{R}^2$

$$\vec{x} = 2\vec{u} + 3\vec{v} - 4\vec{w}$$

Task: find <u>another</u> linear combination of $\vec{u}, \vec{v}, \vec{w}$ that $= \vec{x}$.

How? $\vec{w} = 3\vec{u} + 4\vec{v} \Rightarrow \vec{0} = 3\vec{u} + 4\vec{v} - \vec{w}$

add! $\vec{x} + \vec{0} = 5\vec{u} + 7\vec{v} - 5\vec{w}$

$\vec{x} + 17(\vec{0}) = 53\vec{u} + 71\vec{v} - 21\vec{w} = \vec{x}$

$$2x - 3y + 4z = 0$$

is this subspace linearly independent?

→ not a meaningful question to ask! subspaces have infinitely many vectors

→ instead, we ask if a finite list of vectors is linearly independent

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ -6 \\ 0 \\ 14 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 3 \\ 7 \end{bmatrix} \right\}$$

1. Is this a basis for $\mathbb{R}^4$? no: every basis of $\mathbb{R}^4$ has exactly 4 vectors
2. dimension of their span? 3, because there are 3 LI vectors
3. linearly independent? no!

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \cancel{\begin{bmatrix} 6 \\ -6 \\ 0 \\ 14 \end{bmatrix}}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \cancel{\begin{bmatrix} 0 \\ 3 \\ 3 \\ 7 \end{bmatrix}} \right\}$$

$$\begin{bmatrix} 6 \\ -6 \\ 0 \\ 14 \end{bmatrix} = 6 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + 14 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 3 \\ 3 \\ 7 \end{bmatrix} = 3 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + 7 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

lin independent vectors is 3

Problem 7: High Definition (12 pts)

Suppose $\vec{x}_1, \vec{x}_2, \ldots \vec{x}_{12}$ are 12 non-zero vectors in $\mathbb{R}^7$. Furthermore, suppose:

- $\vec{x}_1$, $\vec{x}_2$, and $\vec{x}_3$ span a 2-dimensional subspace of $\mathbb{R}^7$.
- $\vec{x}_4$, $\vec{x}_5$, and $\vec{x}_6$ span **the same** 2-dimensional subspace of $\mathbb{R}^7$ as $\vec{x}_1$, $\vec{x}_2$, and $\vec{x}_3$, i.e.

$$\text{span}(\{\vec{x}_4, \vec{x}_5, \vec{x}_6\}) = \text{span}(\{\vec{x}_1, \vec{x}_2, \vec{x}_3\})$$

a) (4 pts) Let r be the dimension of the subspace of $\mathbb{R}^7$ spanned by $\vec{x}_1, \vec{x}_2, \ldots \vec{x}_{12}$. What are the smallest and largest possible values of r? Your answers should be integers with no variables.

smallest possible value of $r =$ ☐ largest possible value of $r =$ ☐

b) (4 pts) Which of the following **could** form a basis for $\mathbb{R}^7$? Select all that apply. Blank answers will receive no credit.

- ☐ $\{\vec{x}_7, \vec{x}_8, \vec{x}_9, \vec{x}_{10}, \vec{x}_{11}, \vec{x}_{12}\}$
- ☐ $\{\vec{x}_6, \vec{x}_7, \vec{x}_8, \vec{x}_9, \vec{x}_{10}, \vec{x}_{11}, \vec{x}_{12}\}$
- ☐ $\{\vec{x}_1, \vec{x}_5, \vec{x}_8, \vec{x}_9, \vec{x}_{10}, \vec{x}_{11}, \vec{x}_{12}\}$
- ☐ $\{\vec{x}_1, \vec{x}_2, \vec{x}_5, \vec{x}_9, \vec{x}_{10}, \vec{x}_{11}, \vec{x}_{12}\}$
- ☐ $\{\vec{x}_1, \vec{x}_2, \vec{x}_8, \vec{x}_9, \vec{x}_{10}, \vec{x}_{11}, \vec{x}_{12}\}$

c) (4 pts) Suppose the intersection of $\text{span}(\{\vec{x}_1, \vec{x}_2\})$ and $\text{span}(\{\vec{x}_4, \vec{x}_5\})$ is a line (i.e. a 1-dimensional subspace) in $\mathbb{R}^7$. Which of the following **must** be true? Select all that apply. Blank answers will receive no credit.
Hint: Don't forget the assumptions introduced at the start of the problem.

- ☐ $\vec{x}_2$, $\vec{x}_4$, and $\vec{x}_5$ can all be written as scalar multiples of $\vec{x}_1$.
- ☐ The set $\{\vec{x}_2, \vec{x}_4\}$ is linearly independent.
- ☐ The set $\{\vec{x}_3, \vec{x}_4\}$ is linearly independent.
- ☐ The set $\{\vec{x}_3, \vec{x}_6\}$ is linearly independent.
- ☐ None of the above.

Problem 5: Well, It Depends

Let $\vec{u} = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 4 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 6 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} 1 \\ -1 \\ \alpha \\ 2 \end{bmatrix}$, where $\alpha \in \mathbb{R}$ is a constant.

$\vec{u} = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 4 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 6 \end{bmatrix}$ $\vec{w} = \begin{bmatrix} 1 \\ -1 \\ 3 \\ 2 \end{bmatrix}$

In parts **a)**, **b)**, **c)**, and **d)**, suppose $\boxed{\alpha = 3}$. Fill in the blanks to complete each sentence.

a) The two vectors $\{\vec{w}, \vec{u} - \vec{w}\}$ are ___(i)___, and span a ___(ii)___-dimensional subspace of $\mathbb{R}^4$.

(i) ● linearly independent ○ linearly dependent

(ii) ○ 1 ● 2 ○ 3 ○ 4

$\dim\big(\underbrace{\text{span}(\{\vec{w}, \vec{u} - \vec{w}\})}_{\text{subspace } S}\big) = 2$

b) The two vectors $\{\vec{v}, 2\vec{v}\}$ are ___(i)___, and span a ___(ii)___-dimensional subspace of $\mathbb{R}^4$.

(i) ○ linearly independent ● linearly dependent

(ii) ● 1 ○ 2 ○ 3 ○ 4

c) The three vectors $\{\vec{u}, \vec{v}, \vec{w}\}$ are ___(i)___, and span a ___(ii)___-dimensional subspace of $\mathbb{R}^4$.

(i) ● linearly independent ○ linearly dependent

(ii) ○ 1 ○ 2 ● 3 ○ 4

$\vec{u} = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 4 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 6 \end{bmatrix}$ $\vec{w} = \begin{bmatrix} 1 \\ -1 \\ 3 \\ 2 \end{bmatrix}$

If the answer is "yes", there is a solution for a, b

Q: Is $\vec{w}$ a linear combination of $\vec{u}, \vec{v}$?

A: NO

$$\vec{w} = a\vec{u} + b\vec{v}$$

$$\begin{bmatrix} 1 \\ -1 \\ 3 \\ 2 \end{bmatrix} = a\begin{bmatrix} 2 \\ 0 \\ -1 \\ 4 \end{bmatrix} + b\begin{bmatrix} 3 \\ 1 \\ 0 \\ 6 \end{bmatrix}$$

$1 = 2a + 3b$

$-1 = b \Rightarrow b = -1$

$3 = -a \Rightarrow a = -3$

$2 = 4a + 6b$

$2(-3) + 3(-1) = -9 \neq 1$

so no solution for a, b!

d) The four vectors $\{\vec{u}, \vec{v}, \vec{w}, \vec{w} - \vec{u}\}$ are ---(i)---, and span a ---(ii)----dimensional subspace of $\mathbb{R}^4$.

(i) ◯ linearly independent ◯ linearly dependent

(ii) ◯ 1 ◯ 2 ◯ 3 ◯ 4

Now, suppose $\boxed{\alpha = -2}$. Fill in the blanks to complete each sentence.

e) The three vectors $\{\vec{u}, \vec{v}, \vec{w}\}$ are ---(i)---, and span a ---(ii)----dimensional subspace of $\mathbb{R}^4$.

(i) ◯ linearly independent ◯ linearly dependent

(ii) ◯ 1 ◯ 2 ◯ 3 ◯ 4

f) The four vectors $\{\vec{u}, \vec{v}, \vec{w}, \vec{w} - \vec{u}\}$ are ---(i)---, and span a ---(ii)----dimensional subspace of $\mathbb{R}^4$.

(i) ◯ linearly independent ◯ linearly dependent

(ii) ◯ 1 ◯ 2 ◯ 3 ◯ 4