EECS 245, Spring 2026

LEC 6

Matrices ; Exam Review

not on MT1!

→ Read : Ch. 5.1 – 5.2

Agenda

1. Review problems
2. Introduce matrices
   - Why do we need them?
   - Addition, scalar mult.
   - Multiplication
   - Transpose, identity

matrices (Ch. 5) are not on Midterm 1

Announcements

- Exam tomorrow, 1-3PM in 1690 BBB
- HW 4 + Lab 5 solutions up
- Will post HW 5 after exam
- OH slots available tomorrow morning!

$$R_{sq}(w) = \frac{1}{n}\sum_{i=1}^{n}(y_i - w)^2$$

$$w^* = \bar{y}$$

$$R_{sq}(w^*) = R_{sq}(\bar{y}) = \frac{1}{n}\sum_{i=1}^{n}(y_i - \bar{y})^2$$

$$= \underbrace{\text{variance}}_{\sigma_y^2}$$

$$\log(a) + \log(b) = \log(ab)$$

$$\sum_{i=1}^{n} \log(x_i) = \log(x_1 x_2 \cdots x_n)$$

$$= \log\left(\prod_{i=1}^{n} x_i\right)$$

Homework 4, Problem 4

a) $7x - 3y + 4z = 0$

infinitely many possible pairs

$$\operatorname{span}\left(\left\{\begin{bmatrix}1\\1\\-1\end{bmatrix}, \begin{bmatrix}0\\4\\3\end{bmatrix}\right\}\right)$$

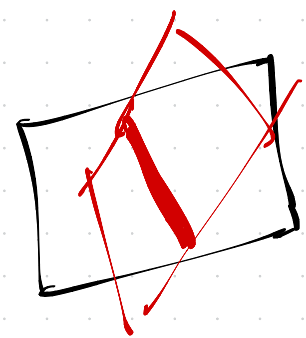
$$= \left\{\begin{bmatrix}x\\y\\z\end{bmatrix} \;\middle|\; 7x - 3y + 4z = 0\right\}$$

b) $\vec{v}_1 = \begin{bmatrix} 7 \\ -1 \\ 2 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ $\vec{v}_3 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$

1. $\text{span}(\{\vec{v}_1, \vec{v}_2\}) : x + y - 3z = 0$
2. $\text{span}(\{\vec{v}_1, \vec{v}_3\}) : -3x - y + 10z = 0$

found using cross product

3. Intersection of planes?



$$x + y - 3z = 0 \quad (1)$$

$$-3x - y + 10z = 0 \quad (2)$$

(1) + (2) $\quad -2x + 7z = 0$

$$7z = 2x$$

$$\frac{7}{2}z = x$$

$$z \begin{bmatrix} 7/2 \\ -1/2 \\ 1 \end{bmatrix}$$

into (1)

$$\frac{7}{2}z + y - \frac{6}{2}z = 0$$

$$\Rightarrow \frac{1}{2}z + y = 0 \quad \Rightarrow y = -\frac{1}{2}z$$

set of solutions

$$= \left\{ z \begin{bmatrix} 7/2 \\ -1/2 \\ 1 \end{bmatrix} \;\middle|\; z \in \mathbb{R} \right\}$$

$$= \left\{ a \begin{bmatrix} 7 \\ -1 \\ 2 \end{bmatrix} \;\middle|\; a \in \mathbb{R} \right\}$$

$$= \operatorname{span}(\{\vec{v}_1\}) \; !$$

parametric form of line / plane

formal: $L = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix} a \;\middle|\; a \in \mathbb{R} \right\}$

same thing

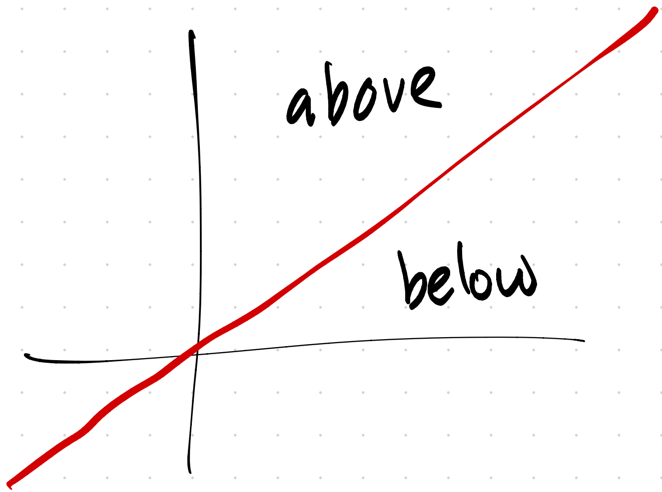
informal: $L = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix} t, \quad t \in \mathbb{R}$

is this the span of a vector?

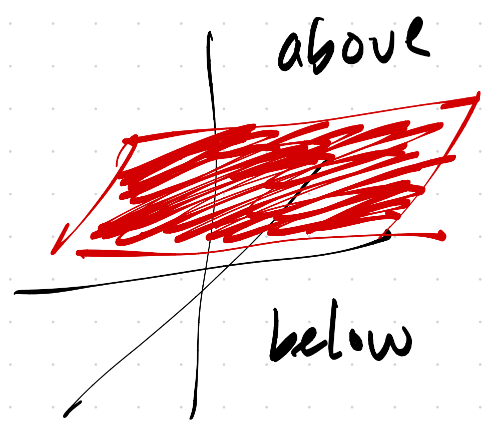
no but if the 3 was a 2, yes

hyperplane : dimension n-1 subspace (or shifted) of $\mathbb{R}^n$

line in $\mathbb{R}^2$



plane in $\mathbb{R}^3$



$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} \pi \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v}_4 = \begin{bmatrix} 13 \\ 0 \\ 0 \\ -2 \end{bmatrix}$, $\vec{v}_5 = \begin{bmatrix} 15 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$\mathrm{span}(\{\vec{v}_1, \ldots, \vec{v}_5\})$ is a $\underline{2}$-dimensional subspace of $\underline{\mathbb{R}^4}$.

linearly independent subset with same span as all 5 : $\{\vec{v}_1, \vec{v}_2\}$

$\{\vec{v}_4, \vec{v}_5\}$

basis for subspace spanned by $\{\vec{v}_1, \ldots, \vec{v}_5\}$

$$a\begin{bmatrix}1\\2\\3\\7\end{bmatrix} + b\begin{bmatrix}3\\2\\7\\4\end{bmatrix} + c\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}3\\7\\2\\4\end{bmatrix}$$

If there exists <u>a</u> solution for a, b, c,

then $\begin{bmatrix}3\\7\\2\\4\end{bmatrix}$ is in

the span of the first 3 vectors

Activity 1: Formal Definition of Linear Independence

Suppose $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d \in \mathbb{R}^n$, and that $\vec{b} \in \text{span}(\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d\})$.

b) Suppose that $a_1\vec{v}_1 + a_2\vec{v}_2 + \ldots + a_d\vec{v}_d = \vec{b}$ **and** $c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_d\vec{v}_d = \vec{b}$, where at least one of the a_i's is different from its corresponding c_i.

Using the formal definition of linear independence from Chapter 4.2, determine whether or not $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d$ are linearly independent, and prove your answer.

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_d\vec{v}_d = \vec{b} \quad (1)$$

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_d\vec{v}_d = \vec{b} \quad (2)$$

$(1) - (2)$

$$(a_1 - c_1)\vec{v}_1 + (a_2 - c_2)\vec{v}_2 + \cdots + (a_d - c_d)\vec{v}_d = \vec{0}$$

at least one $a_i - c_i \neq 0$!

so, $\vec{v}_1, \ldots, \vec{v}_d$ are linearly dependent

Find another set of coefficients $k_1, k_2, \ldots, k_d$ such that

$$k_1\vec{v}_1 + k_2\vec{v}_2 + \ldots + k_d\vec{v}_d = \vec{b}$$

and at least one of the k_i's is different from its corresponding a_i or c_i.

By doing this, you're showing that if there is at least one way to write $\vec{b}$ as a linear combination of a set of vectors, then there are infinitely many ways to write $\vec{b}$ as a linear combination of those vectors; there can't just be two or three ways to do it.

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \ldots + a_d\vec{v}_d = \vec{b} \quad (1)$$

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_d\vec{v}_d = \vec{b} \quad (2)$$

$$\underbrace{(a_1+c_1)}_{2}\vec{v}_1 + \underbrace{(a_2+c_2)}_{2}\vec{v}_2 + \ldots + \underbrace{(a_d+c_d)}_{2}\vec{v}_d = \underbrace{2\vec{b}}_{2}$$

$$A = \begin{bmatrix} 5 & 3 & 5 & 2 \\ 3 & 0 & -6 & 4 \\ -2 & 0 & 4 & 3 \\ 8 & 2 & -6 & -8 \\ 1 & 1 & 3 & 0 \end{bmatrix}$$

4 columns, each of which is a vector in $\mathbb{R}^5$

5 rows, each in $\mathbb{R}^4$

difficulty:

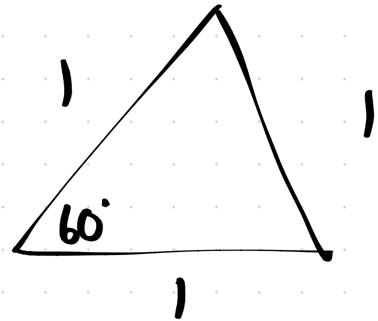
mock < wn26 < fa25

70% 63%

do this yourself tonight!

b) (6 pts) Suppose $\vec{w}, \vec{z} \in \mathbb{R}^n$. Given that $\|\vec{w}\| = \|\vec{z}\| = \|\vec{w} - \vec{z}\| = 1$, find $\|\vec{w} + \vec{z}\|$. Show your work, and $\boxed{\text{circle}}$ your final answer, which should be a number with no variables.

$$\|\vec{w}\| = \|\vec{z}\| = \|\vec{w} - \vec{z}\| = 1$$

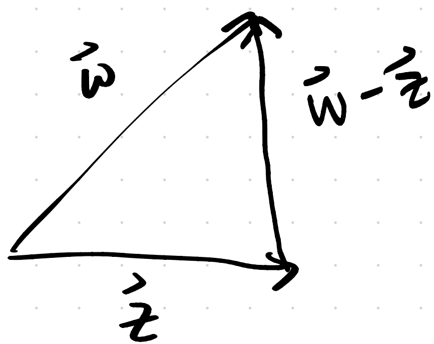


FA25 MT1 4b

$$\|\vec{w} + \vec{z}\| = ?$$

remember:

$$\|\vec{x}\|^2 = \vec{x} \cdot \vec{x}$$



$$\|\vec{w} - \vec{z}\|^2 = 1$$

$$(\vec{w} - \vec{z}) \cdot (\vec{w} - \vec{z}) = 1$$

$$\underbrace{\vec{w} \cdot \vec{w}}_{= 1^2} - 2\vec{w} \cdot \vec{z} + \underbrace{\vec{z} \cdot \vec{z}}_{= 1^2} = 1$$

$$\|\vec{w}-\vec{z}\|^2 = 1$$

$$(\vec{w}-\vec{z})\cdot(\vec{w}-\vec{z}) = 1$$

$$\vec{w}\cdot\vec{w} - 2\vec{w}\cdot\vec{z} + \vec{z}\cdot\vec{z} = 1$$

$$1^2 - 2\vec{w}\cdot\vec{z} + 1^2 = 1$$

$$-2\vec{w}\cdot\vec{z} = 1-2 = -1$$

$$\vec{w}\cdot\vec{z} = 1/2$$

$$\|\vec{w}+\vec{z}\|^2 = (\vec{w}+\vec{z})\cdot(\vec{w}+\vec{z})$$

$$= \vec{w}\cdot\vec{w} + 2\vec{w}\cdot\vec{z} + \vec{z}\cdot\vec{z}$$

$$= 1 + 2\cdot 1/2 + 1 = 3$$

$$\|\vec{w}+\vec{z}\| = \sqrt{3}$$

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos \theta$$

FA25 MT1 P3

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Problem 3: Spreading Your Wings (12 pts)

Consider a dataset of n points, $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, where

- the means of $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ are 15 and 5, respectively
- the variances of $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ are σ_x^2 and σ_y^2, respectively
- the correlation coefficient between $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ is r

We define a new set of values, $z_1, z_2, \ldots, z_n$, as follows:

$$z_i = 3x_i - y_i, \quad i = 1, 2, \ldots, n$$

key: z_i depends on both x_i, y_i!

$\frac{1}{n}\sum z_i$

$= \frac{1}{n}\sum(3x_i - y_i)$

a) (4 pts) Suppose we fit a simple linear regression line to the dataset $(x_1, z_1), (x_2, z_2), \ldots, (x_n, z_n)$ by minimizing mean squared error. Note that z is the variable being predicted, not y. Let $h(x_i)$ represent the corresponding line.

What is the value of $h(15)$? Your answer should be a number with no variables.

$h(15) = \boxed{40}$ = average of z_i's

$z_i = 3x_i - y_i$

$\bar{z} = 3\bar{x} - \bar{y}$

$= 3 \cdot 15 - 5$

$= 40$

b) (8 pts) σ_z^2, the variance of $z_1, z_2, \ldots, z_n$, can be written in the form $\sigma_z^2 = 9\sigma_x^2 + \sigma_y^2 + C$.

(i) What is the value of C?

○ $-6\sigma_x\sigma_y$ ○ $6\sigma_x\sigma_y$ ○ $-6r\sigma_x\sigma_y$ ○ $6r\sigma_x\sigma_y$ ○ $-6nr\sigma_x\sigma_y$ ○ $6nr\sigma_x\sigma_y$

Recall: $\bar{x} = 15$, $\bar{y} = 5$, $z_i = 3x_i - y_i$

$a(x+y+z) = ax + ay + az$

Given: $\sigma_z^2 = 9\sigma_x^2 + \sigma_y^2 + C$

Find C

$$\sigma_z^2 = \frac{1}{n}\sum_{i=1}^{n}(z_i - \bar{z})^2 = \frac{1}{n}\sum_{i=1}^{n}\left(3x_i - y_i - (3\bar{x} - \bar{y})\right)^2$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left(\overset{\square}{3(x_i - \bar{x})} - \overset{\Delta}{(y_i - \bar{y})}\right)^2$$

$\square^2 - 2\square\Delta + \Delta^2$

$$= \frac{1}{n}\sum_{i=1}^{n}\left(9(x_i - \bar{x})^2 - 6(x_i - \bar{x})(y_i - \bar{y}) + (y_i - \bar{y})^2\right)$$

$$= 9\left(\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2\right) - \frac{6}{n}\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}) + \frac{1}{n}\sum_{i=1}^{n}(y_i - \bar{y})^2$$

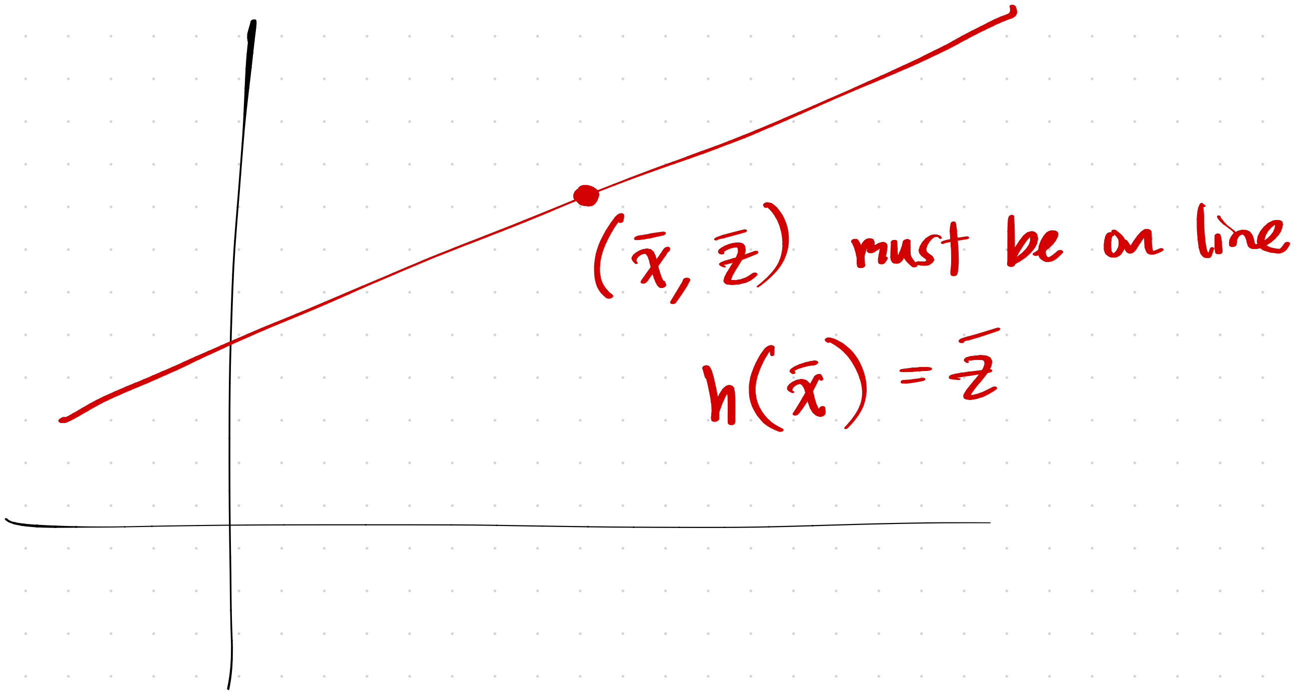
$$= 9\left(\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2\right) - \frac{6}{n}\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y}) + \frac{1}{n}\sum_{i=1}^{n}(y_i-\bar{y})^2$$

$$= 9\sigma_x^2 + \sigma_y^2 - 6(r\sigma_x\sigma_y)$$

Recall:

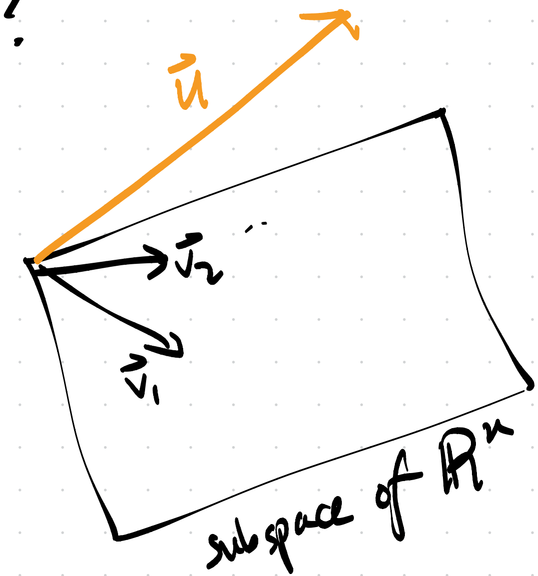
$$r = \frac{1}{n\sigma_x\sigma_y}\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})$$

$$r\sigma_x\sigma_y = \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})$$


$(\bar{x}, \bar{z})$ must be on line!
$h(\bar{x}) = \bar{z}$

New content: matrices

why?



which vector in span$(\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d\})$ is closest to $\vec{u}$?

$\rightarrow$ to answer, we need matrices

Matrix : Rectangular array of numbers

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 2 & 1 & 9 \\ 0 & -1 & 0 \\ 2 & -2 & 0 \end{bmatrix}_{4\times 3}$$

$\vec{v}_1$ $\vec{v}_2$ $\vec{v}_3$

capital letters

4 rows, 3 columns

4×3 "4 by 3"

$A \in \mathbb{R}^{4\times 3}$

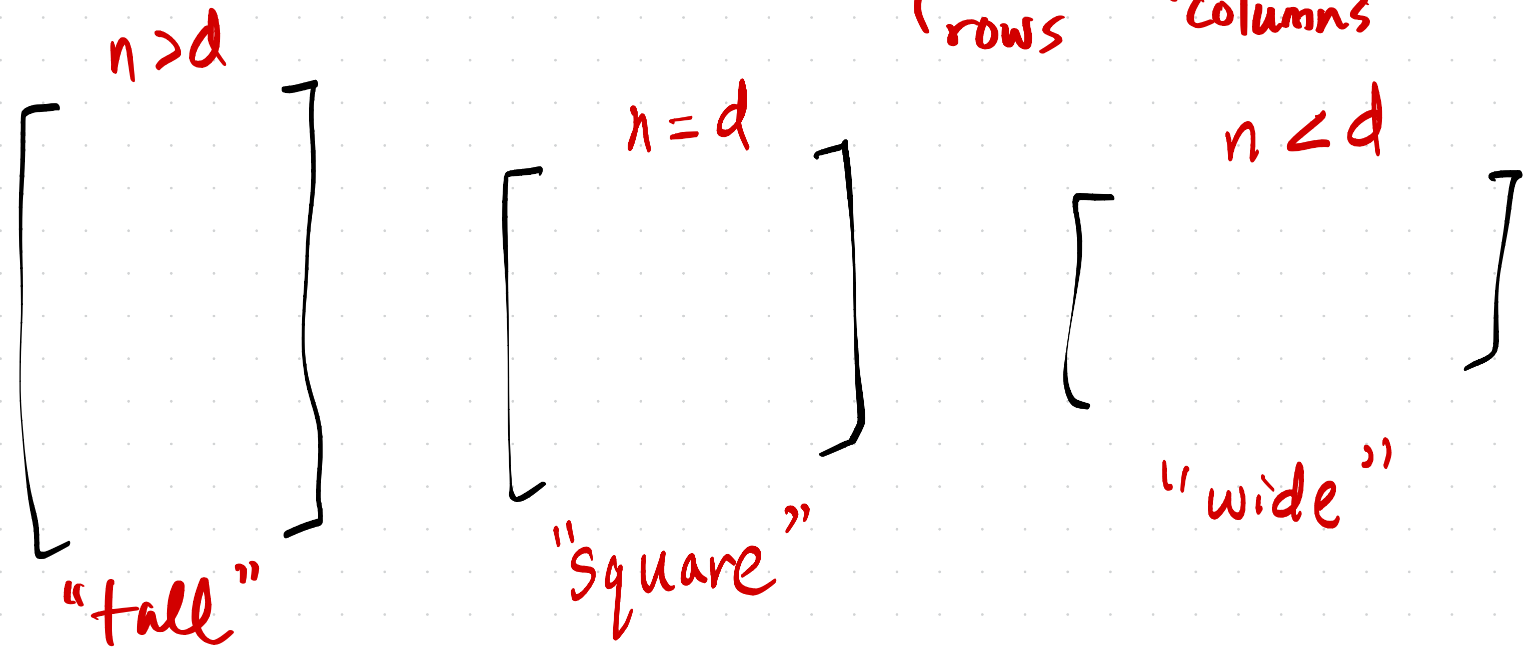
$A_{ij} = a_{ij}$ = row i, column j

$a_{23} = 9$

default interpretation: column vectors written next to each other

In general, A is $n \times d$.

rows columns



Addition and scalar multiplication

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 2 & 1 & 9 \\ 0 & -1 & 0 \\ 2 & -2 & 0 \end{bmatrix}_{4\times 3} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}_{4\times 3}$$

$$3A - B = \begin{bmatrix} & & 9_{1,3} \\ & & \\ & -11_{3,2} & \\ & & \end{bmatrix}_{4\times 3}$$

"Golden Rule" for matrix multiplication

→ In order to be able to multiply

$A_{n \times d}$ and $B_{d \times p}$

inner dim must match!

$\underbrace{\#\text{ columns in } A}_{d} = \underbrace{\#\text{ rows in } B}_{d}$

AB has shape $n \times p$ (outer dimensions)

Matrix-vector multiplication

$$A_{4\times 3} = \begin{bmatrix} 3 & 1 & 4 \\ 2 & 1 & 9 \\ 0 & -1 & 0 \\ -2 & -2 & 0 \end{bmatrix}$$

$\vec{x} \in \mathbb{R}^3$

$$\vec{x}_{3\times 1} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

$$A_{4\times 3}\,\vec{x}_{3\times 1} = \begin{bmatrix} 15 \\ 29 \\ 0 \\ 2 \end{bmatrix}_{4\times 1}$$

~ dot products of $\vec{x}$ with rows of A!!!

$\vec{x}_{3\times 1} A_{4\times 3}$ not legal!

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 2 & 1 & 9 \\ 0 & -1 & 0 \\ 2 & -2 & 0 \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

$$A\vec{x} = 1\begin{bmatrix} 3 \\ 2 \\ 0 \\ 2 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 1 \\ -1 \\ -2 \end{bmatrix} + 3\begin{bmatrix} 4 \\ 9 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 15 \\ 29 \\ 0 \\ 2 \end{bmatrix}$$

Important: $A\vec{x}$ is a linear combination of A's columns with coefficients coming from $\vec{x}$!

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 2 & 1 & 9 \\ 0 & -1 & 0 \\ 2 & -2 & 0 \end{bmatrix}_{4 \times 3} \qquad B = \begin{bmatrix} 1 & 2 \\ 0 & 7 \\ 3 & 2 \end{bmatrix}_{3 \times 2}$$

$$AB = \begin{bmatrix} 15 & 21 \\ 29 & \vdots \\ 0 & -7 \\ 2 & \vdots \end{bmatrix}_{4 \times 2}$$

in general, col j of AB

$= A \times$ col j of B

Properties

- $(AB)C = A(BC)$ ✓ associative
- $(A+B)C = AC+BC$ ✓ distributive
- **Not** commutative! In general,

$$AB \neq BA \,!$$

$A_{n \times d}$ $B_{d \times p}$

AB is a valid product

$B_{d \times p} A_{n \times d}$

just because AB is valid, doesn't mean BA is!

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 0 \end{bmatrix}_{3\times 2} \qquad B = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix}_{2\times 3}$$

AB : output is 3×3

BA : output is 2×2

even if AB and BA both exist, they don't need to have same shape!

what if A, B both square?

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

$$AB = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix} \qquad BA = \begin{bmatrix} 23 & \cdot \\ \cdot & \cdot \end{bmatrix}$$

even then, $AB \neq BA$ in general

Identity matrix

$$I_2 = I_{2\times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$I_{3\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$I_{4\times 4} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

1's on <u>diagonal</u>

$a_{ii} = 1$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & 15 \end{bmatrix}$$

"diagonal"

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I_{2\times 2}} \begin{bmatrix} 5 \\ 3 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

equivalent of multiplying by 1

$$\begin{bmatrix} 7 & \pi & 12 \\ 0 & 0 & 2 \\ 3 & 7 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & \pi & 12 \\ 0 & 0 & 2 \\ 3 & 7 & -2 \end{bmatrix}$$

Activity 4.1 in Ch 5.1

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} 4 \\ 6 \\ 12 \end{bmatrix}$$

$$P\vec{x} = \begin{bmatrix} 12 \\ 4 \\ 6 \end{bmatrix} \qquad D\vec{x} = \begin{bmatrix} 16 \\ 3 \\ 36 \end{bmatrix} \qquad P(D\vec{x}) = \begin{bmatrix} 36 \\ 16 \\ 3 \end{bmatrix} \qquad D(P\vec{x}) = D\begin{bmatrix} 12 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 48 \\ 2 \\ 18 \end{bmatrix}$$