EECS 245, Spring 2026

LEC 7 Rank, Column Space, Null Space

→ Read: Ch. 5.1 - 5.4

→ Watch the supplemental videos on the course website

My freshman year transcript

Fall 2016

Class	Title	Un.	Gr.
CHEM 1A	General Chemistry	3	B-
CHEM 1AL	General Chemistry Laboratory	1	C+
COMPSCI 61A	The Structure and Interpretation of Computer Programs	4	B+
COMPSCI 70	Discrete Mathematics and Probability Theory	4	A
COMPSCI 195	Social Implications of Computer Technology	1	P
MATH 1A	Calculus	4	A+

Kind of like EECS 245

Spring 2017

Class	Title	Un.	Gr.
COMPSCI 61B	Data Structures	4	B+
COMPSCI 97	Field Study	1	P
COMPSCI 197	Field Study	1	P
ELENG 16A	Designing Information Devices and Systems I	4	B-
MATH 110	Linear Algebra	4	C
MATH 128A	Numerical Analysis	4	B+

Math 217

That said, grades do still matter.

The easiest path to a good grade - and more importantly, mastery - is to do the labs and homeworks yourself without AI assistance AND fully understand the solutions

exams are 70% of your grade!

On the bright side, don't forget the redemption policy!

Agenda

Watch the supplemental videos in course website!

- Recap: Matrices (5.1)
- The transpose operator (5.2)
- The "column space" and "rank" of a matrix (5.3)
- The "null space" of a matrix (5.4)
- Basics of invertibility (6.2; this is Thursday's focus)

Announcements

- DM me for exam/other regrades
- Look at "Grade Report" on Gradescope
- Lab 6 due tomorrow
- HW 5 due Thursday
- HW 6 due Sunday
- Come to OH to review your midterm!

Simplify $\|A\vec{x}\|^2 = (A\vec{x}) \cdot (A\vec{x})$

$A_{n \times d}$ columns matrix

rows

$\vec{x}_{d \times 1}$ vector in $\mathbb{R}^d$

$\underbrace{(A\vec{x})_{n \times 1}}$ $\in \mathbb{R}^n$

dot product of every row of A, with $\vec{x}$!

Aside: Transpose

$$A_{3\times 2} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

$A_{n\times d}$

$A^T_{d\times n}$

$$A^T_{2\times 3} = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

when transposing, rows turn into columns, and vice versa!

We can rewrite the dot product using transposes!

$$\vec{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

$$\vec{u} \cdot \vec{v} = \underbrace{\begin{bmatrix} 1 & -1 & 2 \end{bmatrix}}_{\vec{u}^T \; 1\times 3} \underbrace{\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}}_{\vec{v} \; 3\times 1} = \vec{v}^T \vec{u} = \vec{v} \cdot \vec{u}$$

Key takeaway: $\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v}$, but not $\cancel{\vec{u}^T \cdot \vec{v}}$!

dot product is only between 2 vectors in $\mathbb{R}^n$, not a column and a row

Back to the original goal

$$\left(AB\right)^T = B^T A^T$$

remember this!
discussed in notes

$A_{n\times d}$ $\qquad$ $\vec{x}_{d\times 1}$

$$\|A\vec{x}\|^2 = \underbrace{(A\vec{x})}_{\square} \cdot \underbrace{(A\vec{x})}_{\triangle}$$

$$= (A\vec{x})^T (A\vec{x})$$

$$= \underset{1\times d}{\vec{x}^T}\, \underset{d\times n}{A^T}\, \underset{n\times d}{A}\, \underset{d\times 1}{\vec{x}}$$

$\rightarrow$ output is a scalar!

$A_{n \times d}$ $\quad A^T_{d \times n}$

AA^T : $n \times n$ matrix

A^TA : $d \times d$ matrix

$A^T = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 0 & -2 \end{bmatrix}_{2 \times 3}$ $\quad A = \begin{bmatrix} 2 & 3 \\ 0 & 0 \\ 1 & -2 \end{bmatrix}_{3 \times 2}$

$(A^TA)_{2 \times 2}$

$$= \underbrace{\begin{bmatrix} 5 & 4 \\ 4 & 13 \end{bmatrix}_{2 \times 2}}$$

Contains the dot products of all pairs of columns of A!

A^TA "dot product matrix"

If A^TA contains column dot products,
(of A)

AA^T contains the dot products of all pairs of rows!

$$A^TA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\|\vec{v}_1\|^2 = 1$

$\vec{v}_1 \cdot \vec{v}_2 = 0$

what does that tell you about A?

→ A has 2 columns

→ A's columns are orthogonal to each other

→ A's columns are unit vectors

the matrix A^TA is **symmetric**

In general, a matrix B is symmetric if $B^T = B$

row 1 = col 1
row 2 = col 2
⋮

→ only square matrices can be symmetric

→ why is A^TA symmetric, no matter what A is?

→ remember, A^TA contains all pairwise dot products, of A's cols, and dot products are commutative

$$(AB)^T = B^T A^T$$

$$(A^T A)^T$$

$$= A^T (A^T)^T$$

$$= A^T A$$

Thus, $A^T A$ symmetric! So is AA^T.

$$A = \begin{bmatrix} 5 & 3 & 2 \\ 0 & -1 & 1 \\ 3 & 4 & -1 \\ 6 & 2 & 4 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$$

$A\vec{x}$ is just a linear combination of A's cols!

$$A\vec{x} = (2)\begin{bmatrix} 5 \\ 0 \\ 3 \\ 6 \\ 1 \end{bmatrix} + (0)\begin{bmatrix} 3 \\ -1 \\ 4 \\ 2 \\ 0 \end{bmatrix} + (-1)\begin{bmatrix} 2 \\ 1 \\ -1 \\ 4 \\ 1 \end{bmatrix}$$

column space of a matrix A,

$\text{colsp}(A)$, is

- the span of A's columns
- i.e. the set of all linear combinations of A's columns
- i.e. the set of all possible outputs of $A\vec{x}$, for all possible $\vec{x}$'s

$$A_{n\times d} = \begin{bmatrix} | & | & & | \\ \vec{a}^{(1)} & \vec{a}^{(2)} & \cdots & \vec{a}^{(d)} \\ | & | & & | \end{bmatrix}_{n\times d} \qquad \vec{a}^{(i)} \in \mathbb{R}^n$$

$$\operatorname{colsp}(A) = \underbrace{\operatorname{span}\left(\left\{\vec{a}^{(1)}, \vec{a}^{(2)}, \ldots, \vec{a}^{(d)}\right\}\right)}_{\text{subspace of } \mathbb{R}^n}$$

$$A = \begin{bmatrix} 5 & 3 & 2 \\ 0 & -1 & 1 \\ 3 & 4 & -1 \\ 6 & 2 & 4 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\dim(\text{colsp}(A)) = 2$$

cols not linearly independent!

col 1 − col 2 = col 3

$$\text{colsp}(A) = \text{span}\left(\left\{ \begin{bmatrix} 5 \\ 0 \\ 3 \\ 6 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 4 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \\ 4 \\ 1 \end{bmatrix} \right\}\right)$$

$$= \text{span}\left(\underbrace{\{\text{col } 1, \text{col } 2\}}_{\text{basis for colsp}(A)}\right) = \text{span}\left(\underbrace{\{\text{col } 2, \text{col } 3\}}_{\text{basis for colsp}(A)}\right)$$

$$\operatorname{rank}(A) = \text{dimension of } \operatorname{colsp}(A)$$

$$= \#\ \text{linearly independent cols}$$

don't forget this!

Activity: Think of a 3×4 matrix with

(1) rank 1

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

(2) rank 2

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 1 & 2 & 3 & 0 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

(3) rank 3

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & 2 & 3 & 0 \\ 1 & 2 & 0 & 4 \end{bmatrix}$$

(4) rank 4

can't do it!

no subspace of $\mathbb{R}^3$ can be 4-dimensional

$$A = \begin{bmatrix} 5 & 3 & 2 \\ 0 & -1 & 1 \\ 3 & 4 & -1 \\ 6 & 2 & 4 \\ 1 & 0 & 1 \end{bmatrix} \qquad A^T = \begin{bmatrix} 5 & 0 & 3 & 6 & 1 \\ 3 & -1 & 4 & 2 & 0 \\ 2 & 1 & -1 & 4 & 1 \end{bmatrix}$$

$\text{rowsp}(A)$ = span of A's rows!

$= \text{colsp}(A^T)$

"row space"

row space is the set of possible outputs of $A^T\vec{y}$, $\vec{y} \in \mathbb{R}^5$

$$A^T = \begin{bmatrix} 5 & 0 & 3 & 6 & 1 \\ 3 & -1 & 4 & 2 & 0 \\ 2 & 1 & -1 & 4 & 1 \end{bmatrix}$$

$$\vec{a}_1 \quad \vec{a}_2 \quad \vec{a}_3 \quad \vec{a}_4 \quad \vec{a}_5$$

Note: each $\vec{a}_i$ satisfies $x-y-z=0$, so they span a 2-d subspace

Fill in the blanks:

colsp(A^T) is a $\underline{2}$-dimensional subspace of $\underline{\mathbb{R}^3}$.

"row space" of A

$\vec{a}_2, \vec{a}_5$ can create all 5 cols of A^T

Fact: For <u>any</u> matrix A, not just square: <u>all</u> matrices

\# linearly independent cols = \# linearly independent rows

$$\operatorname{rank}(A) = \operatorname{rank}(A^T)$$

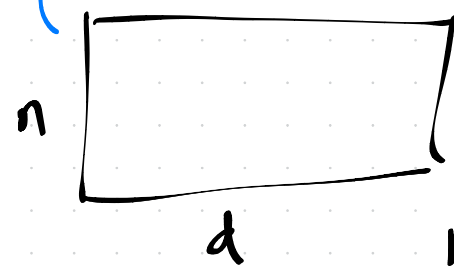
A $n \times d$ matrix

$\mathrm{rank}(A) = \mathrm{rank}(A^T)$

$n > d$

cols could be LI, but rows cannot be LI

cols cannot be LI



$n < d$

If A square, then either

- BOTH the cols and rows are LI
- or NEITHER

n

d

$n = d$

$$A = \begin{bmatrix} 5 & 3 & 2 \\ 0 & -1 & 1 \\ 3 & 4 & -1 \\ 6 & 2 & 4 \\ 1 & 0 & 1 \end{bmatrix}$$

Q: Are all of A's columns linearly independent?

A: No, because

→ col 3 = col 1 − col 2

→ $\vec{0} = 1(\text{col } 1) (-1)\text{col } 2 (-1)\text{col } 3$

$$A \underbrace{\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}}_{\vec{x}} = \vec{0}$$

not in rowsp

not in colsp

The only reason $A\vec{x} = \vec{0}$ had a non-trivial solution for $\vec{x}$ is that A's cols are **NOT** linearly independent!

null space of A

suppose A is $n \times d$

$$\text{nullsp}(A) = \underbrace{\left\{ \vec{x} \in \mathbb{R}^d \;\middle|\; A\vec{x} = \vec{0} \right\}}_{\text{set of all } \vec{x}\text{'s such that } A\vec{x} = \vec{0}}$$

subspace of $\mathbb{R}^d$

→ video

intuition: set of $\vec{x}$'s that get "destroyed" when multiplied by A

key idea: if A's columns are linearly independent, then the only solution to $A\vec{x} = \vec{0}$ is $\underline{\vec{x} = \vec{0}}$.

$\Rightarrow$ if A's cols are linearly independent,

$$\text{nullsp}(A) = \{\vec{0}\}$$ "trivial" null space

New example A:

col 4 = 3 col 1 − 5 col 3

$\to$ 3 col 1 − 5 col 3 − 1 col 4 = $\vec{0}$

$$A = \begin{bmatrix} 3 & 6 & 0 & 9 & 3 \\ 2 & 4 & 0 & 6 & 2 \\ 0 & 0 & 1 & -5 & 0 \\ 1 & 2 & 0 & 3 & 1 \end{bmatrix}$$

Q: what is $\underbrace{\dim(\text{nullsp}(A))}_{\text{subspace of } \mathbb{R}^5} = 3$?

col 2 = 2 col 1

$\vec{0}$ = 2 col 1 − col 2

rank$(A) = 2$

$$A\begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \vec{0} \qquad A\begin{bmatrix} 3 \\ 0 \\ -5 \\ -1 \\ 0 \end{bmatrix} = \vec{0} \qquad A\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} = \vec{0}$$

col 5 = col 1

Rank-nullity theorem

suppose A is an $n \times d$ matrix.

$$\mathrm{rank}(A) + \dim(\mathrm{nullsp}(A)) = \underbrace{d}_{\#\text{ cols in } A}$$

$$A = \begin{bmatrix} 3 & 6 & 0 & 9 & 3 \\ 3 & 4 & 0 & 6 & 2 \\ 2 & 0 & 1 & -5 & 0 \\ 0 & 2 & 0 & 3 & 1 \\ 1 & & & & \end{bmatrix}$$

$\mathrm{rank}(A) = 2$, so

$\dim(\mathrm{nullsp}(A)) = 5 - 2$
$= 3$

$$A = \begin{bmatrix} 5 & 3 & 2 \\ 0 & -1 & 1 \\ 3 & 4 & -1 \\ 6 & 2 & 4 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\operatorname{rank}(A) = 2$$

$$\dim(\operatorname{nullsp}(A)) = 3-2$$
$$= 1$$

$$\operatorname{nullsp}(A) = \operatorname{span}\left(\left\{ \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \right\} \right)$$

nullsp(A) subspace of $\mathbb{R}^3$

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$$
$$\begin{array}{ccc} \vec{b}^{(1)} & \vec{b}^{(2)} & \vec{b}^{(3)} \end{array}$$

① $\operatorname{rank}(B) = 1$

② $\dim(\operatorname{nullsp}(B)) = 2$

③ Find basis for $\operatorname{nullsp}(B)$

one possible ans: $\left\{ \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} \right\}$

$2\vec{b}^{(1)} = \vec{b}^{(2)}$

$\rightarrow 2\vec{b}^{(1)} - 1\vec{b}^{(2)} + 0\vec{b}^{(3)} = \vec{0}$

$3\vec{b}^{(1)} = \vec{b}^{(3)}$

$\rightarrow 3\vec{b}^{(1)} + 0\vec{b}^{(2)} - 1\vec{b}^{(3)} = \vec{0}$

$B \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \vec{0}$!

so $\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \in \operatorname{nullsp}(B)$

$$B = \begin{array}{ccc} & & \\ \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \\ \vec{b}^{(1)} \quad \vec{b}^{(2)} \quad \vec{b}^{(3)} \end{array}$$

(1) $\text{rank}(B) = 1$

(2) $\dim(\text{nullsp}(B)) = 2$

(3) Find basis for $\text{nullsp}(B)$

one possible ans: $\left\{ \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} \right\}$

e.g. $5\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} - 17\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -41 \\ -5 \\ 17 \end{bmatrix}$

also in nullsp!

$$B\begin{bmatrix} -41 \\ -5 \\ 17 \end{bmatrix} = \begin{bmatrix} -41-10+3\cdot 17 \\ -82-20+6\cdot 17 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \vec{0}$$

Example: 7×9 matrix A, $\text{rank}(A) = 5$

Dimensions of .....

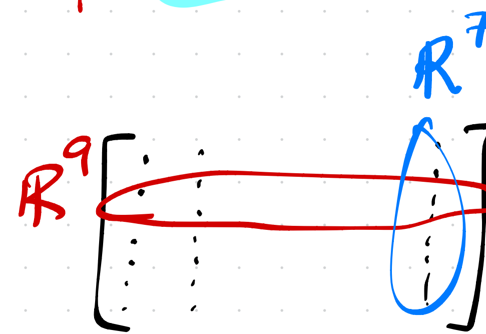
① $\text{colsp}(A)$: $\underline{5}$-dimensional subspace of $\mathbb{R}^{\underline{7}}$

② $\text{colsp}(A^T)$: 5-dimensional subspace of $\mathbb{R}^9$

③ $\text{nullsp}(A)$: 4-dimensional subsp of $\mathbb{R}^9$

"left null space"

④ $\text{nullsp}(A^T)$: 2-dimensional subsp of $\mathbb{R}^7$



$\{\vec{y} \mid A^T\vec{y} = \vec{0}\}$

$$\text{rank}(A) + \dim(\text{nullsp}(A)) = d_{(\#\text{cols of } A)}$$

$$\underbrace{\text{rank}(A^T)}_{\text{rank}(A)} + \dim(\text{nullsp}(A^T)) = n \quad \#\text{cols of } A^T$$

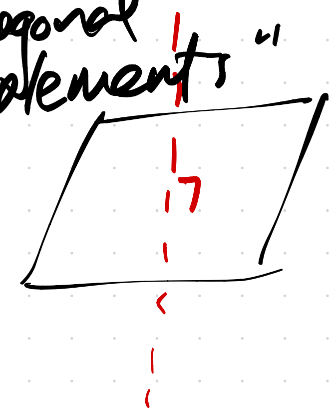
$$A_{n \times d} \qquad {A^T}_{d \times n}$$

Two supplemental videos posted in the linked playlist:

1. $\text{rank}(X) = \text{rank}(X^T X)$

2. rowsp and nul'sp are "orthogonal complements"

   same true for colsp and nullsp(A^T)



Briefly, inverses

scalar addition :

$$5 + (-5) = 0$$

"identity"

adding 0 doesn't change a value!

0^{-1} doesn't exist!

scalar multiplication :

$$7\underbrace{(7)^{-1}}_{1/7} = 1$$

multiplicative identity

Inverse of a matrix (multiplication)

A^{-1} inverse of A

$$\Rightarrow \quad A^{-1}A = AA^{-1} = I = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}$$

multiplying by I doesn't change any values!

$\Rightarrow$ A must be <u>square</u> to have an inverse!

only square matrices can be invertible, but not all are!

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}}_{\text{not invertible!}} \qquad \underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}_{\text{invertible}}$$

invertible, because its columns are linearly independent

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}} = \vec{b}$$

→ can I find $\vec{x}$ such that $\vec{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$?

no, can't be done

→ what about $\vec{b} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$? yes, but ∞ ways