EECS 245, Spring 2026

LEC 12

Diagonalization,

Spectral Theorem,

SVD

→ Read: Ch. 9.4, 9.5, 10.1, 10.2

Agenda

- Recap: eigenvalues, eigenvectors, diagonalization
- Algebraic and geometric multiplicities
- The spectral theorem
- Singular value decomposition: extending to non-square matrices

Announcements

- Remaining deadlines:
  - Lab 11: tomorrow
  - HW 10: Thursday
  - HW 11: Sunday
  - Lab 12: Monday
- Lab 10 solutions posted
- Extra credit: fill out both the End-of-Semester Survey + the Official Evals for 1% EC

Final Exam : Wednesday, June 24th, 8-10 AM,
Room : 1018 DOW

- Exam is cumulative! All content is in scope.
  - ≈ 30% MT1, 30% MT2, 40% new content
  - See redemption policy
- To prep: old Final Exams + a new review worksheet
- Lots of office hours slots – sign up and come with questions!
- Lecture 14 (day before exam) will mostly be review; come with questions


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Examples (taken from 9.4)

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

For each matrix, find all eigenvalues and eigenvectors.
Is it diagonalizable?

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

"default way"

characteristic poly: $\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix}$

$$= (1-\lambda)(4-\lambda) - 4$$

$$(1-\lambda)(4-\lambda) - 4 = 0$$

$$4 - \lambda - 4\lambda + \lambda^2 - 4 = 0$$

$$\lambda^2 - 5\lambda = 0$$

$$\lambda(\lambda - 5) = 0 \quad \Rightarrow \quad \lambda = 0, \quad \lambda = 5$$

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

"easy way"

easy way to spot eigenvalues:

→ since A's cols not independent,

→ $\lambda = 0$ is an eigval!

→ $\lambda_1 + \lambda_2 = \text{trace}(A) = 1 + 4 = 5$

→ $\lambda_1 \lambda_2 = \det(A) = 0$

→ $\lambda = 5$ other eigval!

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

$\lambda_1 = 0$

$A\vec{v}_1 = \vec{0}$ $\qquad \vec{v}_1 = \begin{bmatrix} a \\ b \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$a + 2b = 0 \Rightarrow a = -2b$

$2a + 4b = 0$ $\qquad$ e.g. $b = 1$, $a = -2$

$\Rightarrow \boxed{\vec{v}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}}$ (could have picked $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 20 \\ -10 \end{bmatrix}$)

$\lambda_2 = 5$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 5a \\ 5b \end{bmatrix}$$

$a + 2b = 5a$

$2b = 4a$

$b = 2a$

e.g. $a = 1$, $b = 2$

$\boxed{\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}}$

Q: Is $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ diagonalizable?

If A is diagonalizable, it means we can write

$$A = V \Lambda V^{-1}$$

V may or may not be invertible!

"eigenvalue decomposition"

cols of V are eigvectors of A

diagonal of Λ:

$$\begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \end{bmatrix}$$

& Review where this came from (see 9.4)

$A = V \Lambda V^{-1}$

① invertibility not related to diagonalizability

$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

② if A not invertible, A has eigval of 0

Diagonalizable? Yes!!!

$V = \begin{bmatrix} -2 & 10 \\ 1 & 20 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}$

→ Multiply $V \Lambda V^{-1}$, you get A!

One application of diagonalizing:

$$A = V\Lambda V^{-1}$$

Matrix powers! Much more efficient than repeated multiplication

$$A^3 = V\Lambda \underbrace{V^{-1} V}_{I} \Lambda \underbrace{V^{-1} V}_{I} \Lambda V^{-1}$$

$$= V \Lambda \Lambda \Lambda V^{-1}$$

$$= V \Lambda^3 V^{-1}$$

$$\begin{bmatrix} \lambda_1^3 & & 0 \\ & \lambda_2^3 & \\ 0 & & \ddots \end{bmatrix}$$

$B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ triangular matrix: eigvals are on diagonal

→ B has $\lambda = 1$ repeated

→ algebraic multiplicity of $\lambda = 1$ is 2 : $AM(1) = 2$

$p(\lambda) = (1-\lambda)^2 = 0$

eigvecs?

$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1a \\ 1b \end{bmatrix} \Rightarrow \begin{matrix} a + b = a \\ b = b \end{matrix}$

solution: $b = 0$, but a can be anything!

$B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$\vec{v} = \begin{bmatrix} 17 \\ 0 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$

all eigvecs for $\lambda = 1$

Is B diagonalizable?

$B = V \Lambda V^{-1}$

$V = \begin{bmatrix} 17 & 2 \\ 0 & 0 \end{bmatrix}$ not invertible!

there is no second linearly independent eigvec!

want to make V with eigvecs of B as columns

$\Rightarrow$ thus, B not diagonalizable!

If a matrix has a repeated eigenvalue, can it be diagonalizable?

$$p(\lambda) = (1-\lambda)^2$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\lambda = 1$ algebraic multiplicity 2

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \Rightarrow \begin{matrix} a = a \\ b = b \end{matrix}$$

Both a and b free,

so all of $\mathbb{R}^2$ is an eigvec!

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$V = \begin{bmatrix} \pi & -2 \\ 17 & 4 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad V^{-1}$$

$$A = V \Lambda V^{-1}$$

Multiplicities

Suppose A is an $n \times n$ matrix with characteristic poly

$$p(\lambda) = (\lambda-\lambda_1)^{m_1}(\lambda-\lambda_2)^{m_2}\cdots(\lambda-\lambda_k)^{m_k}$$

degree n poly

$m_1 + m_2 + \cdots + m_k = n$

- The algebraic multiplicity of λ_i is

$$AM(\lambda_i) = m_i$$

number of times λ_i is a root of $p(\lambda)$

- the algebraic multiplicities alone don't tell us whether A is diagonalizable.

9.4, Activity 2

Suppose an $n \times n$ matrix A has the characteristic polynomial

$$p(\lambda) = (\lambda+1)^2 \lambda^1 (\lambda-1)^3 (\lambda-4)^2 (\lambda-5)^1 (\lambda-12)^2$$

1. What is n (i.e. the number of rows/columns of A)? 11
2. What is the determinant of A? 0 : $\lambda = 0$ is an eigval, so $\det(A)$ $=$ prod of λ's $= 0$
3. What are all of A's eigenvalues and their algebraic multiplicities?
4. Is it diagonalizable?

no way to tell! need to know more

λ	$AM(\lambda)$
-1	2
0	1
1	3
$\vdots$	$\vdots$

is diagonalizable!

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ~ eigvecs are all of $\mathbb{R}^2$

$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is not! ~ eigvecs all lived on a line, spanned by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

<u>Both</u> have same $p(\lambda) = (1-\lambda)^2$

Both have $AM(1) = 2$, but the geometric multiplicity, $GM(1)$, is different for both

$GM(\lambda_i)$ = dimension of the eigenspace for λ_i

= dimension of the subspace of all eigenvectors for eigenvalue λ_i

$$= \dim\left(\underbrace{\mathrm{nullsp}(A - \lambda_i I)}_{\text{eigenspace for } \lambda_i}\right)$$

Aside: suppose $\vec{v}$ is an eigenvector of A for $\lambda = 7$

$$A\vec{v} = 7\vec{v}$$

$$A\vec{v} - 7I\vec{v} = \vec{0}$$

$$(A - 7I)\vec{v} = \vec{0}$$

$$\Rightarrow \vec{v} \in \text{nullsp}(A - 7I)$$

For any λ_i of a matrix A,
eigenvalue

$$\underbrace{AM(\lambda_i)}_{\text{potential dimension}} \geq \underbrace{GM(\lambda_i)}_{\text{actual dimension of eigenspace for } \lambda_i} \geq 1$$

For A to be diagonalizable, we need

$$AM(\lambda_i) = GM(\lambda_i)$$

for all eigenvalues λ_i of A.

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

① Find all eigvals and multiplicities

② Is it diagonalizable?

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

"block diagonal":
eigenvalues of big matrix are eigenvalues of individual blocks

$\lambda = 3, \lambda = -1$ $\qquad$ $\lambda = 3$

→ First, find λ_i's

→ $p(\lambda) = \begin{vmatrix} 1-\lambda & 2 & 0 \\ 2 & 1-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix}$

$$= (1-\lambda)\big((1-\lambda)(3-\lambda)\big) - 2\big(2(3-\lambda)\big) + 0$$

$$= (3-\lambda)\big((1-\lambda)^2 - 4\big)$$

$$= (3-\lambda)(\lambda+1)(\lambda-3)$$

$\lambda = 3$, $AM(3) = 2$

$\lambda = -1$, $AM(-1) = 1$

Is $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ diagonalizable?

→ All we need to check is if $GM(3) = 2$

→ How do we find $GM(3)$?

$$A - 3I = \begin{bmatrix} -2 & 2 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\dim(\text{nullsp}(A-3I)) = 2$, since $\text{rank}(A-3I) = 1$

$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $\quad \vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

eigenspace for $\lambda = 3$

$= \text{nullsp}(A - 3I)$

$= \text{span}\left(\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}\right)$

$$\underbrace{\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}}_{A} \begin{bmatrix} -7 \\ -7 \\ 12 \end{bmatrix} = \begin{bmatrix} -21 \\ -21 \\ 36 \end{bmatrix} \checkmark$$

So, e.g. $-7 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 12 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -7 \\ -7 \\ 12 \end{bmatrix}$ must be an eigenvector of A with $\lambda = 3$!

Back to the Q.

Is $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ diagonalizable?

Yes!

$\lambda = 3$: $AM = GM = 2$ ✓

$\lambda = -1$: $AM = GM = 1$ ✓

$$B = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$AM(2) = 2$
$AM(3) = 1$

We need to find $GM(2)$

$$B - 2I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow$ rank$(B-2I) = 2$
$\Rightarrow \dim(\text{nullsp}(B-2I)) = 1$

so, bad news!
$GM(2) = 1$,

only one LI eigenvec for $\lambda = 2$

$$\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

so, since $GM(2) < AM(2)$,
B <u>not</u> diagonalizable

"Spectral theorem": symmetric matrices are special!

Suppose A is an $n \times n$ symmetric matrix, with real numbers as entries. Then, the following hold:

① A has n real-valued eigenvalues, factoring in algebraic multiplicities

② Eigenvectors for different eigenvalues are orthogonal!

③ For every eigenvalue λ_i, $AM(\lambda_i) = GM(\lambda_i)$

①+②+③: Any symmetric matrix A can be diagonalized by an orthogonal matrix Q:

$$A = V \Lambda V^{-1} \Rightarrow A = Q \Lambda Q^T$$

A symmetric : $A = Q \Lambda Q^T$ exists

$$Q = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix}$$

→ spectral theorem guarantees $\vec{v}_i$'s are automatically orthogonal if corresponding to different λ's

→ also, each $\vec{v}_i$ must be a unit vector

cols are still eigvecs of A

→ if they correspond to the same λ, choose orthogonal vecs in that eigenspace

Gram-Schmidt →

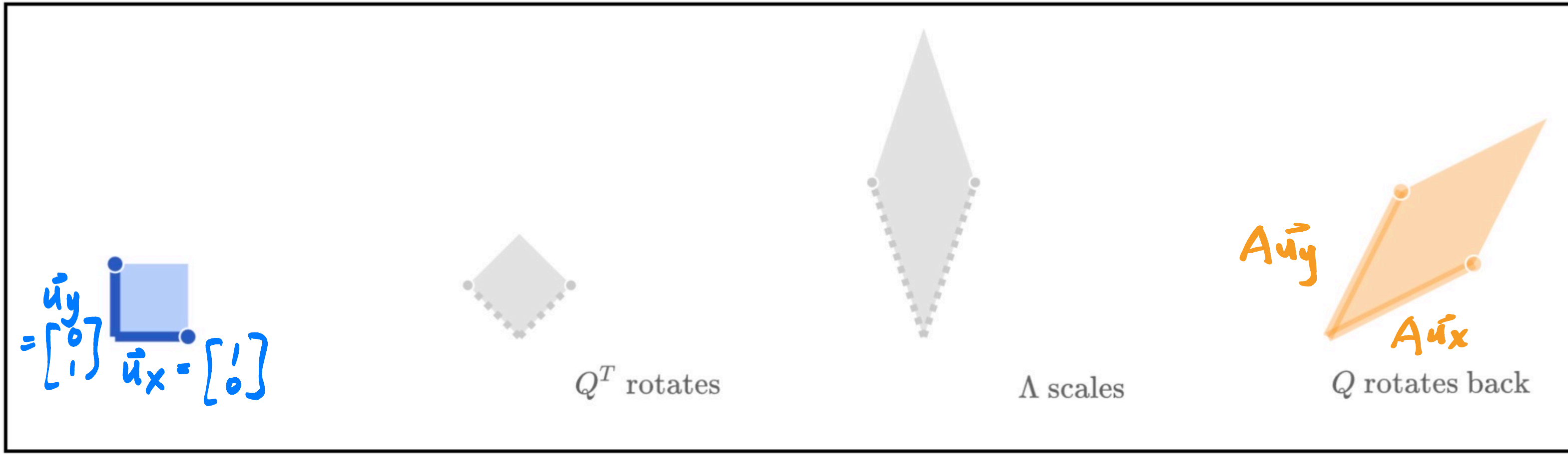
$$A = Q \Lambda Q^T$$

Q orthogonal : $Q^T Q = Q Q^T = I$

$$Q^T = Q^{-1}$$

Q^T is easier to find than Q^{-1} in general

Visualizing $A = Q\Lambda Q^T$



$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = Q \Lambda Q^T$$

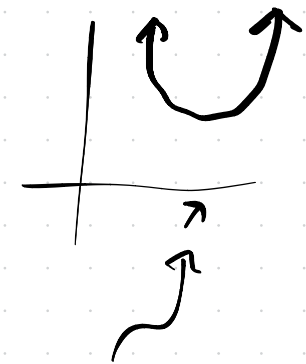
$$A\vec{x} = Q\Lambda Q^T \vec{x}$$

Study: what is the result of $Q^T\vec{x}$ in this context?

Aside: what matrix doesn't have real-valued eigenvalues?

→ e.g. rotation matrix, which is an orthogonal matrix

$$A^TA = I$$

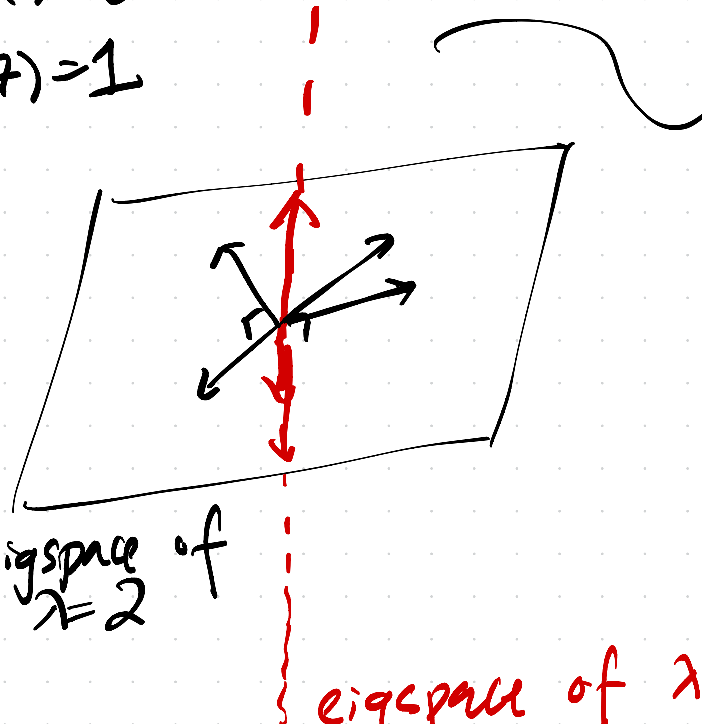


→ e.g. $A = \begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix}$

$$\det(A - \lambda I) = \text{- - - - -} \qquad \lambda^2 + C$$

$\lambda = 2, \ GM(2) = 2$

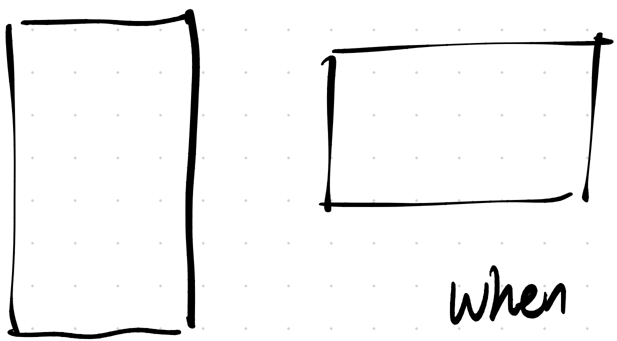
$\lambda = 7, \ GM(7) = 1$



spectral theorem guarantees eigenvecs for different eigvals are orthogonal

Final topic: singular value decomposition

Ch. 10, HW 11, Lab 12



How does all this relate to non-square matrices?

When we first introduced eigenvalues and eigenvectors, the key equation was

$$\underset{n\times n}{A}\,\underset{n\times 1}{\vec{v}} = \lambda \underset{n\times 1}{\vec{v}}$$

$\vec{v}$ and $A\vec{v}$ both in $\mathbb{R}^n$

Suppose X is an $n \times d$ matrix

left singular vector

$$X_{n\times d} \, \vec{v}_{i\,d\times 1} = \sigma_i \, \vec{u}_{i\,n\times 1}$$

$\vec{v} \in \mathbb{R}^d$

$\vec{u} \in \mathbb{R}^n$

"right" singular vector

constant, called "singular value"

Think of $(\vec{u}_i, \vec{v}_i, \sigma_i)$ as a triplet of friends

Singular value decomposition

Exists for every $n \times d$ matrix X

U orthogonal!

$U^T U = U U^T$
$= I_{n \times n}$

columns of U are the left singular vectors, $\vec{u}_i$

$$X_{n \times d} = U_{n \times n} \Sigma_{n \times d} V^T_{d \times d}$$

V orthogonal!

$V^T V = \cancel{V V^T} = I_{d \times d}$

cols of V are the $\vec{v}_i$'s, right singular vectors

?

not necessarily square!

Σ : diagonal matrix with σ_i's on diagonal

$$\begin{bmatrix} \sigma_1 & & & 0 \\ & \sigma_2 & & \\ 0 & & \ddots & \sigma_r \\ & & 0 & \end{bmatrix}$$

Important:

$\sigma_1 \geq \sigma_2 \geq \dots \sigma_r > 0$

order from big to small

$$X = \begin{bmatrix} 3 & 2 & 5 \\ 2 & 3 & 5 \\ 2 & -2 & 0 \\ 5 & 5 & 10 \end{bmatrix}$$

remember: rows of V^T are cols of V

$$X = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{3\sqrt{2}} & -\frac{1}{\sqrt{3}} & -\frac{2}{3} \\ \frac{1}{\sqrt{6}} & -\frac{1}{3\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{2}{3} \\ 0 & \frac{2\sqrt{2}}{3} & 0 & \frac{1}{3} \\ \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} & 0 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 15 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{bmatrix}}_{V^T}$$

$$X \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix} = 15 \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ 0 \\ 2/\sqrt{6} \end{bmatrix}$$

$$X\vec{v}_1 = \sigma_1 \vec{v}_1$$

Where did the SVD come from?

$$X = U \Sigma V^T$$

→ X is not square ---- X^TX and XX^T are both square <u>and</u> symmetric!

→ short answer: cols of V are eigvecs of X^TX

cols of U are eigvecs of XX^T

$\sigma_i = \sqrt{\lambda_i \text{ of } X^TX}$