Lab 7: Rank, Column Space, Null Space, and Inverses

EECS 245, Winter 2026 at the University of Michigan

due by the end of your lab section

Name: ____________________

uniqname: ____________________

Each lab worksheet will contain several activities, some of which will involve writing code and others that will involve writing math on paper. To receive credit for a lab, you must complete all activities and show your lab TA by the end of the lab section.

While you must get checked off by your lab TA **individually**, we encourage you to form groups with 1-2 other students to complete the activities together.

Recap: Rank, Column Space, and Null Space (Chapter 5.3 and 5.4)

Suppose A is an $n \times d$ matrix. Then, rank(A) is the number of linearly independent columns of A.

	Notation	**Description**	**Dimension**	**Subspace of**
Column space	colsp(A)	Span of the columns of A	rank(A)	$\mathbb{R}^n$
Row space	colsp(A^T)	Span of the rows of A	rank(A)	$\mathbb{R}^d$
Null space	nullsp(A)	Set of all vectors $\vec{x}$ such that $A\vec{x} = \vec{0}$	$d -$ rank(A)	$\mathbb{R}^d$

Additionally, note that you can write the dot product of two vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$ as $\vec{u}^T\vec{v}$:

$$\vec{u}^T = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

$$\vec{u}^T\vec{v} = u_1v_1 + \cdots + u_nv_n = \sum_{i=1}^{n}(u_iv_i) = \vec{u} \cdot \vec{v} \qquad \textbf{(not } \vec{u}^T \cdot \vec{v}\textbf{)}$$

Activity 1: Null Space of a Matrix with Linearly Independent Columns

Let $A = \begin{bmatrix} 3 & 0 \\ 0 & 4 \\ 1 & 0 \end{bmatrix}$. What is nullsp$(A)$?

Activity 2: Fundamentals

Let $X = \begin{bmatrix} 1 & 2 & -1 & 3 & 4 & 4 \\ 2 & 5 & -2 & 7 & 11 & 10 \\ 4 & 8 & -4 & 12 & 16 & 16 \end{bmatrix}$.

a) Find a basis for $\text{colsp}(X)$. What is $\text{rank}(X)$? Why? *Hint: Column 5 is a linear combination of columns 1 and 2. With this fact, you should be able to answer this relatively quickly.*

b) Fill in the blanks: $\text{colsp}(X^T)$ is a ____-dimensional subspace of ____.

c) Fill in the blanks: $\text{nullsp}(X)$ is a ____-dimensional subspace of ____.

d) Find a basis for $\text{nullsp}(X)$. *Hint: You should be able to answer this without solving equations.*

Suppose A is an $n \times d$ matrix with rank r. A CR decomposition of A is a product of two matrices C and R, where $A = CR$ and:

- C is an $n \times r$ matrix and R is a $r \times d$ matrix
- C contains the linearly independent columns of A, selected left-to-right
- R tells how to "mix" the columns of C (which are linearly independent) to reconstruct the columns of A

Let's keep working with $X = \begin{bmatrix} 1 & 2 & -1 & 3 & 4 & 4 \\ 2 & 5 & -2 & 7 & 11 & 10 \\ 4 & 8 & -4 & 12 & 16 & 16 \end{bmatrix}$.

e) Find a CR decomposition of X. This shouldn't take very much work; review your work from part **a)** in finding a basis for colsp(X).

The key idea being assessed here is that in $A = CR$, the columns of C are linearly independent and a basis for colsp(A), while the rows of R are linearly independent and a basis for colsp(A^T).

Activity 3: Outer Products

Suppose $A = \vec{u}\vec{v}^T + \vec{w}\vec{z}^T$, where $\vec{u}, \vec{v}, \vec{w}, \vec{z} \in \mathbb{R}^n$ are non-zero vectors.

a) What is rank$(\vec{u}\vec{v}^T)$?

b) Under what conditions is rank$(A) = 2$? What about rank$(A) < 2$? *Hint: First, think about what happens when multiplying A by a vector $\vec{x} \in \mathbb{R}^n$. Can you write this as a linear combination of some other vectors? The case for rank$(A) = 2$ is more complicated than "$\vec{u}$ and $\vec{w}$ are linearly independent."*

Activity 4: The Systems of Equations View

Let A be an $n \times d$ matrix of rank r, and suppose there exists vectors $\vec{b} \in \mathbb{R}^n$ such that

$$A\vec{x} = \vec{b}$$

does not have a solution (meaning no $\vec{x}$ makes $A\vec{x} = \vec{b}$).

a) What are all inequalities ($<$ or $\leq$) that must be true between n, d, and r?

b) How do you know that $A^T\vec{y} = \vec{0}$ has solutions other than $\vec{y} = \vec{0}$?

The lab after Spring Break will cover inverses in more detail. For now, let's get our feet wet. Refer to **Chapter 6.1** and **Chapter 6.2** for the relevant definitions and theorems.

Activity 5: Symbolic Inverses

Given that A is an invertible $n \times n$ matrix that satisfies $A^4 - 3A^2 + 2A - 4I = 0$, find an expression for A^{-1} in terms of A.

Activity 6: Basics of Invertibility

Suppose A is an $n \times n$ matrix. State as many of the equivalent conditions for invertibility as you can.

Activity 7: Programming

Complete the tasks in the `lab07.ipynb` notebook, which you can either access through the DataHub link on the course homepage or by pulling our GitHub repository. To receive credit for Activity 7, you'll need to show your lab TA that all test cases have passed. Instructions on how to do this are in the lab notebook.