Lab 11: Eigenvalues and Eigenvectors, Convexity

EECS 245, Winter 2026 at the University of Michigan

due by the end of your lab section

Name: ______________________________

uniqname: ______________________________

Each lab worksheet will contain several activities, some of which will involve writing code and others that will involve writing math on paper. To receive credit for a lab, you must complete all activities and show your lab TA by the end of the lab section.

While you must get checked off by your lab TA **individually**, we encourage you to form groups with 1-2 other students to complete the activities together.

Recap: Eigenvalues and Eigenvectors

Let $A = \begin{bmatrix} 6 & 3 \\ 3 & -2 \end{bmatrix}$.

- An **eigenvector** of A is a non-zero vector $\vec{v}$ such that $A\vec{v} = \lambda\vec{v}$ for some scalar λ. The scalar λ is called the **eigenvalue** corresponding to $\vec{v}$. For A's eigenvectors, multiplying by A is equivalent to multiplying by a scalar.
- The **characteristic polynomial** of A is given by $p(\lambda) = \det(A - \lambda I)$.

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 6-\lambda & 3 \\ 3 & -2-\lambda \end{vmatrix} = (6-\lambda)(-2-\lambda) - 3 \cdot 3 = \lambda^2 - 4\lambda - 21 = (\lambda+3)(\lambda-7)$$

- The eigenvalues of A are the roots of the characteristic polynomial, so $\lambda_1 = -3$ and $\lambda_2 = 7$.
  - The eigenvector $\vec{v}_1$ satisfies $A\vec{v}_1 = -3\vec{v}_1$.

$$\begin{bmatrix} 6 & 3 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = -3 \begin{bmatrix} a \\ b \end{bmatrix} \implies b = -3a$$

    So any vector of the form $\begin{bmatrix} a \\ -3a \end{bmatrix}$ ($a \neq 0$) is an eigenvector of A corresponding to the eigenvalue -3. We could pick $\boxed{\vec{v}_1 = \begin{bmatrix} 2 \\ -6 \end{bmatrix}}$.
  - The eigenvector $\vec{v}_2$ satisfies $A\vec{v}_2 = 7\vec{v}_2$. Another way to find it is to solve for the null space of $A - 7I = \begin{bmatrix} -1 & 3 \\ 3 & -9 \end{bmatrix}$. One vector in $\text{nullsp}(A - 7I)$ is $\boxed{\vec{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}}$.

Activity 1

For each 2×2 matrix A below:

(i) Find the characteristic polynomial of A, and use it to find the eigenvalues of A.

(ii) Find one eigenvector for each eigenvalue of A. Verify that each eigenvector is indeed an eigenvector of A by multiplying it by A.

(iii) **By hand (not using Python or Desmos)**, draw a picture (like the one in Chapter 9.1 titled **Visualizing the eigenvectors of A**) with vectors $\vec{v}_1, A\vec{v}_1, \vec{v}_2, A\vec{v}_2$ as arrows (where $\vec{v}_1$ and $\vec{v}_2$ are the eigenvectors you found above).

a) $A = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$

b) $A = \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix}$

Activity 2: Rapid Fire

The goal of this activity is to practice spotting eigenvalues and characteristic polynomials quickly. Two quick facts:

- The **sum** of the eigenvalues of a matrix is equal to the **trace** of the matrix (which is the sum of the diagonal entries).
- The **product** of the eigenvalues of a matrix is equal to the **determinant** of the matrix.

a) A 2×2 matrix A has trace$(A) = 5$ and det$(A) = 6$. What are the eigenvalues of A?

b) A non-invertible 2×2 matrix has an eigenvalue of 5. What is its characteristic polynomial?

c) A 3×3 matrix A has det$(A) = 20$ and two unique **positive integer** eigenvalues, one of which is repeated twice. In other words, $p(\lambda)$ has the form

$$p(\lambda) = (\lambda - \lambda_1)^2(\lambda - \lambda_2)$$

(λ_1 has an **algebraic multiplicity** of 2. This is a term we'll see more next week.)
What are all possible values of λ_1 and λ_2?

Activity 3: Quadratic Forms Return

Open Desmos in 3D mode at **desmos.com/3d** and write $z = x^2 + 2bxy + 16y^2$. This should show you a 3D surface along with a slider for b. Drag the slider to see how the shape of the surface changes for different b's. You should notice that depending on the value of b, the surface may or may not have a global minimum. Let's explore!

a) z is a quadratic form, $f(\vec{x}) = \vec{x}^T A\vec{x}$, where $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and A is a symmetric matrix. Find A.

b) For a vector-to-scalar function $f : \mathbb{R}^n \to \mathbb{R}$, the **Hessian** of f, denoted $\nabla^2 f$, is the $n \times n$ matrix of second partial derivatives of f. Find $\nabla^2 f$ for $f(\vec{x}) = \vec{x}^T A\vec{x}$.

c) A symmetric matrix A is **positive semidefinite** (PSD) if $\vec{v}^T A\vec{v} \geq 0$ for all $\vec{v} \in \mathbb{R}^n$. In English, this says that A is positive semidefinite if the quadratic form $f(\vec{v}) = \vec{v}^T A\vec{v}$ is always non-negative for all $\vec{v} \in \mathbb{R}^n$. Two relevant facts:

- A differentiable vector-to-scalar function f is **convex** if its Hessian is PSD.
- A symmetric matrix A is PSD if and only if all of its eigenvalues are non-negative.

Using the facts above, find the range of values b for which f is convex, and verify your answer by dragging the slider on Desmos.

Activity 4: Understanding Complex Proofs

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function. It turns out that the function $g(\vec{x})$, defined by

$$g(\vec{x}) = f(A\vec{x} + \vec{b})$$

for some $n \times n$ matrix A and vector $\vec{b} \in \mathbb{R}^n$, is also convex, no matter what A and $\vec{b}$ are. We're not going to ask you to prove this on your own: instead, we'll give you a proof and ask you questions to ensure you understand it.

Our **goal** is to show that $g((1-t)\vec{x} + t\vec{y}) \leq (1-t)g(\vec{x}) + tg(\vec{y})$, for all $\vec{x}, \vec{y} \in \mathbb{R}^n$ and $t \in [0, 1]$. We'll start with the "left-hand side" of the definition, and try and leverage f's convexity.

$$\begin{aligned} g((1-t)\vec{x} + t\vec{y}) &= f\left(A\left((1-t)\vec{x} + t\vec{y}\right) + \vec{b}\right) && (1) \\ &= f\left((1-t)A\vec{x} + tA\vec{y} + \vec{b}\right) && (2) \\ &= f\left((1-t)(A\vec{x} + \vec{b}) + t(A\vec{y} + \vec{b})\right) && (3) \\ &\leq (1-t)f(A\vec{x} + \vec{b}) + tf(A\vec{y} + \vec{b}) && (4) \\ &= \boxed{(1-t)g(\vec{x}) + tg(\vec{y})} && (5) \end{aligned}$$

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a) In which line did we use the fact that f is convex?

b) How did we move from line (1) to line (2), i.e. $f\left(A\left((1-t)\vec{x} + t\vec{y}\right) + \vec{b}\right) = f\left((1-t)A\vec{x} + tA\vec{y} + \vec{b}\right)$?

c) How did we move from line (2) to line (3), i.e. $f\left((1-t)A\vec{x} + tA\vec{y} + \vec{b}\right) = f\left((1-t)(A\vec{x} + \vec{b}) + t(A\vec{y} + \vec{b})\right)$?

Recall, $g(\vec{x}) = f(A\vec{x} + \vec{b})$, where A is an $n \times n$ matrix and $\vec{x}, \vec{b} \in \mathbb{R}^n$. On the last page, we showed that if f is convex, then g is convex.

Now, let's explore what happens if f is **strictly** convex. Recall, this means that for all (non-equal) $\vec{x}$ and $\vec{y}$ in its domain, and for any $t \in (0, 1)$,

$$f((1-t)\vec{x} + t\vec{y}) < (1-t)f(\vec{x}) + tf(\vec{y})$$

d) Suppose $\text{rank}(A) = n$. Explain why it's impossible for $A\vec{x} + \vec{b} = A\vec{y} + \vec{b}$ for two different vectors $\vec{x}$ and $\vec{y}$.

e) Suppose $\text{rank}(A) < n$. Explain why it's possible for $g(\vec{x}) = g(\vec{y})$ for two different vectors $\vec{x}$ and $\vec{y}$. *Hint: Think about nullsp*(A).

f) Using the above reasoning, explain why if f is strictly convex, then g is strictly convex if $\text{rank}(A) = n$, and is (not strictly) convex if $\text{rank}(A) < n$.

g) What were your thoughts on this type of activity, where we give you a proof and ask you questions about it?

○ Hated it ○ Didn't like it ○ Neutral ○ Liked it ○ Loved it